The Influence of Lesson Design on the Teaching and Learning Process in Secondary School Mathematics.

Michael G. Rickhuss

A thesis submitted for the degree of
Doctor of Philosophy

University of Wolverhampton
2019

The Influence of Lesson Design on the Teaching and Learning Process in Secondary School Mathematics

MICHAEL G. RICKHUSS MPhil, BSc (Hons)

A thesis submitted in partial fulfilment of the requirements of the University of Wolverhampton for the degree of Doctor of Philosophy

March 2019

This work or any part thereof has not previously been presented in any form to the University or to any other body whether for the purposes of assessment, publication or for any other purpose (unless otherwise indicated). Save for any express acknowledgments, references and/or bibliographies cited in the work, I confirm that the intellectual content of the work is the result of my own efforts and of no other person.

The right of Michael George Rickhuss to be identified as author of this work is asserted in accordance with ss.77 and 78 of the Copyright, Designs and Patents Act 1988. At this date copyright is owned by the author.

Signature *M. G. Rickhuss*

Date *2nd March 2019*

Abstract

This study involves a collaborative qualitative case study in a single secondary school with two classes of 11 and 12 year old pupils and a group of mathematics teachers. Its aim was to investigate the use of pedagogical language and terminology relating to lesson design for the teaching of mathematics. The rationale was based on a critical review of the literature arguing that pedagogical terms are often used interchangeably by teachers and that these have particular meanings when designing lessons and for the learning of mathematics.

The research was viewed through the lens of a single study school with a group of teachers who had all received their initial teacher training in the same institution (the one in which I work). The research process involved the plan-do-review cycles during which the participating teacher facilitated the video recording of the lessons with the classes. Each of these lessons was followed by a conversation in the form of semi-structured interviews between the teachers and researcher supported by video recordings of classroom interactions.

Following each analysis and evaluation of the lessons the participating teachers had time and space to develop lesson plans using their newly acquired understandings of the pedagogical terminology. The thesis outlines the ways in which the project developed through the cycles. The conversations between teachers and researcher were analysed using a form of analysis based on dialogic assumptions about the multi-voiced nature of talk. The findings suggest that there were changes in the ways in which the teachers communicated with each other about their ideas of lesson design. Pupil interview data suggests that children experienced an increased opportunity to explore an aspect of mathematics. Pupils also developed a deeper conceptual understanding of what is a mathematical abstract concept (the division of fractions), and that this was independent of prior attainment.

Although the findings do suggest a shift in teacher use and understanding of pedagogical terminology relating to lesson design, there were issues around using small groups of pupils and a single setting for generalisation but not for transferability to other mathematical topics. The study does conclude that there is a strong link between teacher shared understandings of pedagogical terminology and lesson planning with the result being pupils from across the attainment range being able to access a mathematically difficult topic.

Finally, it is acknowledged that there are multiple demands being placed upon practising teachers attempting to implement a myriad of changes together with the approaches from this research. Even given these multiple constraints their enthusiasm and learning resulted in changes to lesson design and a common shared understanding of terminology for the framing of lessons.

Glossary

Authenticity: A task is authentic if it matches a situation found in the real world or patterns of interaction are similar to those found in the real world.

Booster Course / Subject Knowledge Enhancement Course (SKE): Funded courses usually to improve subject knowledge prior to entering ITT typically of eight weeks duration. Originally introduced by the TDA and now run by the DfE.

Closed Task: A task that requires learners to reach a single correct solution.

Cognitive Complexity: The extent to which the cognitive operations required to perform a task are easy or difficult.

Collaborative Dialogue: The talk that enables learners to perform a task.

Consciousness-raising Tasks: A task that engages the learner in thinking and communicating.

Co-operative Learning: Learning that results from group work and engages the learners in collaborative dialogue. Each learner adds to or extends the learning of others.

CPD: Continuing professional development

Declarative Knowledge: Declarative Knowledge is characterised by Anderson (1996) as 'knowledge that'. In the case of mathematics it consists of factual information.

Focused Task: A task that has been designed to induce learners' attention to some specific mathematics when processing information.

Implicit Learning: Learning that takes place without awareness.

Informational-Gap Task: A task where one person holds information that others do not have.

ITT: Initial teacher training.

Learning Episode: A period of time during a lesson when learning takes place, independent of the type of learning pedagogy employed.

NQT: Newly qualified teacher

O*f*STED: Office for Standards in Education, Children's Services and Skills – originally established in 1992 as the Office for Standards in Education. In 2001 it was renamed when its responsibilities were expanded to include inspection of day care and childminding in England.

One-way Task: An informational task where one person holds all the information to be communicated.

Opinion-gap Tasks: A task that requires the learners to exchange opinions. Such tasks involve controversial issues where learners may hold different views.

Open Task: A task where the learners know there is no predetermined single solution.

Pedagogic Tasks: Tasks designed to elicit communication in the classroom. These may not bear a resemblance to real world tasks.

PCK/SKfT: Pedagogic content knowledge /subject knowledge for teaching – both names refer to the same set of concepts

PCK: Pedagogic Content Knowledge

PGCE: Postgraduate Certificate of Education

Procedural Knowledge: Knowledge that is easily and rapidly accessible during the performance of a task.

Reasoning Task: A task that requires participants to engage in reasoning, synthesising information and deducing new facts. Prabhu (1987) distinguishes reasoning tasks from information-gap and opinion-gap tasks.

Schematic Knowledge: This consists of 'schemata' mental structures for organising different knowledge.

SKfT: Subject knowledge for teaching

STEM: Science, Technology, Engineering and Mathematics

Structured Task: A task that lends itself to learners using a ready-made schema. A structured task is less complex than an unstructured task.

Subject Mentors: Support individual mathematics teacher trainees in schools to develop their PCK/SKfT. They are also responsible for the assessment of the trainee.

Target Task: A task found in the real-world.

Task Complexity: The extent to which a particular task is easy or difficult.

Task Difficulty: The extent to which a learner finds a task easy or difficult. Individual learner factors (intelligence, learning style, motivation) are responsibility of task difficulty.

Task Procedures: The methodological strategies used to teach the task.

Teach First: Employment-based route of ITT based on Teach for America whereby highly qualified graduates are employed by state schools in low income, hard to recruit areas following an intensive six-week induction programme.

TIMSS: Trends in Mathematics and Science Study

TTA: Teacher Training Agency - former name for the UK government's teacher training agency 1993 – 2005

Two-way Tasks: A task where information exchange is divided between two or more learners.

Unfocused Task: A task designed to encourage comprehension and understanding.

Years 7, 8, 9, 10, 11, 12, 13: UK Secondary School years and US grade equivalents

Year 7 (age 11-12) = 6th grade
Year 8 (age 12-13) = 7th grade
Year 9 (age 12-14) = 8th grade
Year 10 (age 14-15) = 9th grade
Year 11 (age 15-16) = 10th grade
Year 12 (age 16-17) = 11th grade
Year 13 (age 17-18) = 12th grade

Zone of Proximal Development (ZPD): A socio-cultural theory to explain how learners involved with a task interact to perform functions that they would be incapable of performing independently. It refers to a learner's potential rather than to an actual level of development.

Contents

List of Figures, Diagrams and Charts

Acknowledgements

I would like to firstly express my thanks to the teachers and children who worked closely with me. Without their commitment this project would not have been possible. Their views and professional working relationships allowed us to laugh and despair at various points along the way. I have learned so much about collaborative work from this experience. I am also grateful to the children from the classes who enthusiastically embraced classroom dialogue, the lessons and for allowing me to intrude into their school community with a video camera.

I wish to formally acknowledge the assistance from several sources during the course of writing this thesis. My two main supervisors Professor Christine Hockings and Dr Angela Gault have supported guided and challenged my thinking. In particular, I would like to thank Professor Christine Hockings who has been an outstandingly stunning supervisor throughout. She has guided, cajoled and pointed me along the pathway to the completion of this thesis. I am also extremely grateful to Christine who has invested unreasonable amounts of time and effort to help me with various aspects of my work. Christine has whole heartedly believed in me and this study and has always been a source of knowledge, challenge and a source of deep probing questions to make me think. Without Christine this study would have infinitely more difficult and possibly would not have been completed. Finally I would like to thanks Dr Brendan Bartram who has latterly been my director of studies for the encouragement and support given having taken over the position in the final stages of this study.

There are, of course, many others who have helped with various aspects of this work through suggestions and discussions. Although I am not going to list them all, they too deserve my thanks. I must however pay thanks to the school its pupils and mathematics teachers in which the research was under taken.

A special tribute to my long suffering wife Maxine, who has encouraged and supported me on what has been a fascinating journey!

Chapter 1 – Introduction

1.1 Introduction

Chapter one is written to provide an overview of this research study and indicate the nature, direction and focus of the work. Firstly, I will provide the motivation as to why I decided to undertake this research and as to why I think it is needed. Secondly, I will state the research questions. Thirdly, I will define the gap, as I perceive it, in current thinking, practice and pedagogy which have motivated me, as a researcher, to conduct this study. Finally I will present a general overview of the study.

I joined the University of Wolverhampton in 2008 as a senior lecturer in mathematics teacher education. This was my first academic post after a long career in the classroom and consultancy positions in the public sector. I had built up a body of knowledge and experience about teaching mathematics, but as a new lecturer I needed to refresh and broaden my theoretical knowledge to deliver the courses I had been employed to support. I began reading the recent literature on learning theories and pedagogy. Very early into the new post I was part of a research cluster, and was encouraged to enrol on a PhD course. This seemed like the ideal opportunity to bring together the experiences I had gained during my career with the recent thoughts expressed in the literature. I envisaged this would be for the benefit of the new cohorts of mathematics teachers I would be training and ultimately the pupils they were teaching.

1.1.1 The Motivation for the Study

My background as a member of the teaching profession for over 40 years, is that of a head of mathematics in a large comprehensive school, a local authority adviser and teacher trainer. Latterly supporting and working alongside qualified teachers, when they are implementing new pedagogical initiatives, it has become apparent that colleagues often use pedagogical terminology in different ways. What has also become evident to me is that colleagues working together can use phrases relating to aspects of lesson design in different ways, yet they appear to comprehend each other's meanings. Having then observed a large number of lessons the resulting manifestations of these phrases can often be very different and diverse. This made me think about whether colleagues really do share a common pedagogical language when they are discussing and designing lessons. This, what I call, fuzzy use of language for discussing, designing and implementing lessons is in complete contrast to the formal precise language they use when talking about the mathematical content. Therefore, over the years I have become acutely aware that there may be a need for a more precise definition of some of the words and phrases we use when designing and implementing mathematics lessons.

My initial plan for this PhD research was to be a single case study, based in a partnership secondary school mathematics department, focusing on just the terminology used in mathematics lesson design. However, it soon became apparent that lesson terminology is intimately bound with learning, teaching and the beliefs of teachers, and therefore it might be impossible to gain sufficient deep insights using a single case study of artefacts and events. After discussions with supervisors, and reference to the literature, action research methodology was considered and disregarded in favour of a qualitative case study approach which would involve artefacts, events, teachers, pupils and a study lesson. Using a case study methodological approach with a group of participating teachers, in their classrooms with their pupils might allow me to gain deep insights into the practices of teachers and the resulting impact on their pupils. Additionally I might also be able to influence, or even change, the current practice of the teachers who had willingly volunteered to be part of the research.

Mathematics teachers, in my experience, often teach replicating the way in which they were taught. This invariably tends to be from a didactic stance where the mathematical content is demonstrated and then practised by the pupil. Teachers who adopt this approach also often believe that the mathematical content has a predominantly hierarchical structure which necessitates pupils learning the basic concepts before moving to more complex ideas. As a means of changing approaches to learning mathematics, I believed that pupils not only need to practice skills but also generate their own ideas, pose their own questions and develop solutions so as to construct their own knowledge and understanding of the subject (Hatch and Gardner, 1990). It is my belief that this approach should be evident and transparent in the pedagogical language used to describe and define lessons. It was from this standpoint that I decided to undertake this research. As a direct consequence of the research and my previous experiences I was also motivated to try to influence the practice of teachers and by inference the experiences for learners.

Over the latter part of my career, whilst working alongside many teachers in their classrooms, I have come to the belief that there is significant evidence in the literature to support my claims that:

> Fixed conceptions of the approaches to the teaching of mathematics can limit the achievements and attainment of pupils and can limit their opportunities to investigate mathematics (Burton, 2012).
>
> Alternative approaches to mathematics based on social constructivism theories could encourage deeper engagement with learning of the subject and empower pupils (Buerk, 1990; Wilson, 2017).

My thoughts were that if I am to influence the next generation of teachers, and by inference their pupils, then I felt undertaking this doctoral research would give me this authority and the opportunity to probe and explore practices. The outcome might be that of informing and hopefully improving the situation for all those involved. I also anticipated that the findings from the doctoral research would be useful for my everyday work as an initial teacher trainer.

Whilst working full-time and also studying, the motivation needed to continue should not be underestimated. At times during the initial stages of reviewing the literature, clarifying the research questions, and trying to set up the fieldwork I did feel demoralised and significantly challenged. The desire to get on with the fieldwork was thwarted at a number of stages, however, once the ethical approval (appendix 37) had been granted and the pilot study had got underway then the self-belief and motivation returned. The fieldwork with pupils and teachers further generated increased motivation and the desire to complete the study and write up this thesis. It was obvious from the teachers involved that the research would improve the situation for pupils sand improve their learning of mathematics. This was by far the principal motivator for completing this research study, but I also recognised that the contribution to knowledge was still an underlying principal aim.

1.1.2 My Research Position

A researcher's positionality is grounded in an individual's views and beliefs, and as Sikes (2004) argues, is therefore understood through their ontological and epistemological assumptions. Therefore, positionality "reflects the position that the researcher has chosen to adopt within a given research study" (Savin-Baden and Major, 2013, p. 71) and is normally identified by locating the researcher in relation to three areas: the subject, the participants and the research context and process (Ibid, p. 71). Little research in the educational field is, or can be, value free (Carr, 2000) and the subjective - contextual aspects that a researcher brings to their work need to be acknowledged and explained. Self-reflection and reflexivity allow the researcher to clearly identify, construct and critique their positionality (Cohen et al., 2013).

Additionally, as identified by Foote and Bartell (2011) positionality can impact on all aspects and stages of the research process

> The positionality that researchers bring to their work, and the personal experiences through which positionality is shaped, may influence what researchers may bring to research encounters, their choice of processes, and their interpretation of outcomes (p. 46).

with Sikes (2004) identifying that

> it is important for all researchers to spend some time thinking about how they are paradigmatically and philosophically positioned and for them to be aware of how their positioning … and the fundamental assumptions they hold – might influence their research related thinking and practice (p. 15).

I bring an insiders' perspective (see my biography section 4.3.6) to this research having trained and worked in the area of mathematics education, however, in respect of the research school context I am obviously considered to be an outsider. My accumulated knowledge, beliefs, values and biases are acknowledged at the outset and throughout this piece of research. The formal mathematical training I received would place me firmly as a positivist researcher, yet this study is clearly founded in an interpretive paradigm where the research aims to promote an understanding of effective teaching and learning practices rather than to prove a theory.

In addition the relationships between me and the participating teachers might impact on my positionality. However, my insider position does nevertheless allow me to place the participating teachers' narratives in contexts that might not be available to someone who is an outsider. Pinnegar and Daynes (2007) suggest that the interactive nature of qualitative research and the professional connections between participating teachers and me aids objectivity and my researcher positionality. I therefore take the position that this research needs to be with, rather than on, the participating teachers (Clandinin and Rosiek, 2007; Guishard et al., 2018). Finally my positionality will shape and influence the interpretation, and ultimately the 'truthfulnesss' of the research. Consequently an open and honest disclosure of a researcher's positionality is an absolute necessity to allow the reader to make informed judgements as to the factors that have influenced the integrity of the research.

1.2 Research Questions

Having reached the view that there was possibly a problem with teachers having a lack of shared understanding of some pedagogical terminology, I felt that the most appropriate approach to researching this area might be through reflective and questioning strategies. Undertaking the research in a supportive school context, alongside and in conjunction with teachers, by observing and influencing their practice also seemed to me to provide a suitable methodological basis for this research.

For pupils to build conceptual understandings of mathematical topics, for example the arithmetical operations on fractions, in order that they can easily apply them to new contexts or problems I believe that teachers need a clear view as to the appropriate pedagogical approaches. This clear view, expressed in a collectively well understood pedagogical language, would enable learners to effectively make meaningful connections and links between topics. Learning which is designed by teachers needs to be based on a clear set of principles that enables the professionals to converse effectively so as to help learners to make mathematical connections between topics.

So the aim of this research study is to explore secondary school mathematics pedagogy with a view to understanding:

1. **What are the influences of lesson design on pupil learning?**
2. **What are the implications of a change of approach to teaching fractions for teacher training?**

with an additional two subsidiary research aims:

What types of learning episodes could support better pupil understanding of mathematics?

and

What apparent mathematical misconceptions and barriers prevent pupil progress?

1.3 Rationale for the Study

As a practising mathematics teacher and university teacher trainer it is frustrating to see school pupils and trainee student teachers learning to recall information and facts rather than having a deep understanding of the mathematical concepts. Research studies such as Harel and Sowder (1998, 2007), and Flores (2002, 2006) whilst investigating students' conceptions of mathematical proof concluded that teaching and learning of the topic incorporates a range of factors including epistemological, cognitive, instructional and social factors. The findings from these studies fascinated me and highlighted what I thought were the main issues when teaching and learning fractions. These studies resonated with my concerns, providing me with the insights that could suggest why pupils have problems when learning about and applying conceptual ideas that relate to fractions. Knowledge and manipulation of fractions is frequently taught as a set of algorithmic, procedural rules or even tricks. This often leads to a learner having a limited degree of instrumental understanding rather than the deeper, relational understanding needed to link mathematical topics together (Skemp, 2006).

Reforms in mathematics education have focused on pupils gaining deeper understandings of mathematical ideas, relations, and concepts rather than focusing just on accuracy and acquisition of skills (Stevenson, 2010). Pang (2000, 2001, 2002) highlights the fact that although many teachers want to promote these aims and goals when focusing on conceptual understanding, reform-oriented teachers do not always understand curriculum reorganisations or are unsuccessful in implementing new initiatives in their classrooms. Teachers in McClain and Cobb's (2001) study felt frustrated with the traditional textbook, didactic view of teaching of mathematics but had difficulty in finding pedagogical approaches that met the needs of students and that were supportive of the curriculum change agenda. There is a tension between being a reformer or an innovator of the curriculum, and the sincerely held belief of how mathematics should be delivered. This tension still exists (Bokhove and Jones, 2014; Foster and Inglis, 2018) given that OfSted (2009, p. 21) had previously highlighted the problem that “pupils simply completed exercises in textbooks or worksheets, replicating the steps necessary to answer questions in National Curriculum tests or external examinations”.

This tension and viewpoint is easily recognisable and I would argue subscribed to by the vast majority of mathematics teachers. Certainly as a practising teacher I struggled to effectively implement pedagogical approaches that were consistent with, and supportive of, the reforms to improve cognitive development and understanding of demanding mathematical ideas and abstract concepts. According to Pang (2000, 2001, 2002, 2005) many teachers’ personal preferred learning approach is in direct contrast or even opposition to those suggested by reforms as being the optimal pedagogies. This leads to teachers not always being able to effectively stimulate, in pupils, a deep conceptual understanding of mathematics because of the tension between their preferred teaching approach and the preferred learning style of pupils. The proficiency of being able to justify a mathematical answer can help teachers comprehend their pupils' understanding of mathematical concepts. It is therefore important for mathematics teachers to continuously evaluate their pedagogical approaches and knowledge of teaching practices. Additionally they need to critically evaluate the curriculum reforms, especially those that promote pupil engagement in conceptual mathematical thinking and the, verbalising of mathematical justifications (Yackel and Cobb, 1996; Hershkowitz and Schwarz, 1999; Pang, 2000; Kazemi and Stipek, 2001; McClain and Cobb, 2001; Yackel, 2002). It is also necessary for teachers to understand how pupils justify their mathematical thinking and ideas for themselves, their peers, and their teachers (Harel and Sowder, 1998; Flores, 2002) and for teachers to be able to effectively communicate this understanding to other professionals.

It is from this viewpoint that the rationale for the undertaking of this study is viewed. Therefore the two main objectives of the study were to

> inform the next generation of mathematics teachers that there are alternate ways of teaching,
>
> demonstrate that an effective, shared pedagogical language is important when designing lessons and learning experiences for pupils. It is important for teachers to have collective shared understanding of terminology to aid in conversations and when sharing learning with pupils.

1.3.1 The Study

Consistent claims of poor teaching and learning in mathematics by politicians, academics, employers, Ofsted and parents motivated me to investigate the links between pedagogical knowledge, understanding, learning tasks, and cognitively challenging mathematical topics. If the claims are to be believed then I wondered if lesson design might be the missing link that teachers need to consider. A central theme, therefore, of the thesis is the consideration of the influence of lesson design upon the way in which pupils learn mathematics and make connections between topics.

A large body of research has shown the advantages of open investigative approaches to the teaching and learning of mathematics (Resnick, 1990; Maher, 1991; Sigurdson and Olson, 1992). The research of Adhami, Johnson and Shayer (1998) in "Thinking Maths Lessons" focussed on open-ended challenges or tasks, with a real life context, rather than the more conventional closed tasks presented through instructional didactic teaching approaches. Shayer and Adhami (2006) eventually concluded that "for the theory and practice of maths teaching itself, one can imagine ahead, in the spectrum of teachers' skills a seamless integration of instructional teaching aimed at increasing children's competence in what they already understand" (p. 105), thus exposing the links between pedagogical knowledge, lesson design and cognitively challenging mathematical concepts. My aim is to examine and explore the ways in which a different approach to lesson designs can influence pupils' learning of mathematics. In order to achieve this I have reviewed the appropriate literature that articulates the nature of knowledge, understanding and lesson design. I have then applied the outcomes of this literature search to assess, through a qualitative case study methodology (O'Brien, 2001; McNiff, 2013), how learners respond to different levels of problem scaffolding during a number of learning episodes within a single lesson.

1.3.2 Research Assumptions

One of the research assumptions was that this group of newly qualified participating teachers and their pupils would be responsive and willing to be involved. While this was a cautious assumption I had previously worked with all participating parties. I also found that working with a pilot group of pupils and their teacher to be informative and a useful guide for the main study. The pilot group consisted of an intentional sample of pupils and a single newly qualified teacher who was known to me. I also made it absolutely clear to all the participating teachers in the study that their answers would be kept anonymous and confidential so that they would feel comfortable and could be honest and open with their views. Lastly, I assumed that participating teachers would have useful things to share about the research, even though their comments would often be based on only one experience. This group of post graduate teachers, all having been trained by the same institution, would also provide a collective perception on the teaching of mathematics that may agree or disagree with the literature. Possibly more importantly this group of participating teachers had been recently exposed to the views of their experienced departmental colleagues.

1.3.3 Why Focus on Fractions

From my experiences over the last ten or so years when interviewing graduates who wish to train as teachers of mathematics, the vast majority are unable to generate a word problem for a mixed number divided by a fraction ($1\frac{3}{4} \div \frac{1}{2}$). Yet they are able to solve the problem by successfully recalling and using an algorithmic procedure that they had been taught in school. Borko et al. (1992) and Ma (1999) reported exactly the same observations and little seems to have changed in the intervening years from 1992 to the present day. Many teachers of mathematics have difficulty in describing the distinctions between fractions, ratio and proportion; yet these are often taught together (Leinhardt and Smith, 1985; Fuller, 1997; Jitendra et al., 2009). Rational number knowledge based in familiar contexts can be a rich source for the exploration of mathematics (Kieren, 1993), and the fundamental importance of fractions for the development of proportional reasoning cannot be under-estimated (Clarke, Roche and Mitchell, 2008). Hence, I decided that fractions might be a rich source of conversations, not only about the mathematics, but also about the pedagogical processes that might be used when teaching the topic.

1.3.4 Why Key Stage 3 Secondary School Pupils?

The selection of years 7 and 8 (where pupils are aged between 11 and 12) for this research study is very deliberate as the understanding of fractions and the associated manipulative skills are developed around this age. Fractions are a central component of both the late primary (ages 10 to 11) and early secondary (ages 11 to 12) in the United Kingdom National Curriculum mathematics programmes of study (DfE, 2013). Additionally these are the key years when the manipulation and arithmetical operations begin to dominate the study of fractions. Whilst I have been training teachers to teach pupils between the ages of 11 and 16 the position has both given me the required detailed knowledge of the Key Stage 3 mathematics curriculum, and more importantly the access to a range and diversity of classrooms when supporting the trainee teachers. This detailed knowledge of the curriculum and access to classrooms were considered to be two important reasons when thinking about and designing this study.

1.3.5 The Pilot School and Study School

I decided to use just one school to pilot the ideas, materials and data collection tools that would be eventually used for the actual study. There were two overriding reasons for this decision;

1. The strong reciprocal partnership that existed between the school and the university
2. The support and wishes of the leadership, teaching staff and pupils of the school to engage and contribute to educational research.

Westbrook Specialist College (not the real name) in the West Midlands is a relatively large, 11-16 college, with approximately 1000 students on roll. The mathematics provision is staffed by predominantly young, newly qualified teachers, the majority of whom do not have mathematics degrees. The departmental rationale for the teaching of mathematics is based on a mixture of teaching resources, styles and approaches with individual teachers given the freedom and licence to experiment and try new approaches. The department is housed in a new, purpose built school located in a largely white, working class area of the West Midlands. This school was used for both the pilot and main research study, with groups of year 7 (11-12 year old) pupils from two different of cohorts. More detail will be provided on the teachers and school in chapter 4.

In the pilot study a group of six self-selecting pupils, all from a single year 7 mathematics teaching group, were presented with a suite of scenarios and materials all relating to the division of fractions. The 40 minute lesson was video recorded for transcription later. The six pupils were all working at the national curriculum level 5-6 (this was above the national average at the time of the study) where these levels are recognised as the “approximate boundary between concrete and abstract thinking” (Thompson, 2003, p. 70). Piaget (1973) identified this boundary as the divide between concrete operational and the final fourth formal operational abstract thinking stage of intellectual development.

The main research classroom study was undertaken during the next academic year with two year 7 teaching sets, detailed profiles are in appendix 15. The one teaching set was working at national curriculum level 5 and the other working levels 2, 3 and 4 (which was below or just at the national average at the time of the study).

1.3.6 Framework for the Study

Thinking of a research framework in terms of an organised set of statements to specify the relationships and describe phenomena (Fain, 2004) is a helpful way in giving a clear view of the boundaries for the research. Research defined in the form of a ‘theoretical framework’ or ‘conceptual framework’, terms which are often used interchangeably (Parahoo, 2006) are the concepts, assumptions, expectations, beliefs, and theories that support and inform your research. Miles and Huberman (1994) remind us that the research assumptions, expectations and beliefs need to be obvious and transparent. They further explain that a framework needs to be “either graphical or narrative form, the main things to be studied—the key factors, concepts, or variables — and the presumed relationships among them” (ibid, p. 18).

It might also be argued that because of the interchangeable nature of the terminology (‘theoretical framework’ and ‘conceptual framework’) confusion in the mind of the researcher might occur and that these ideas are of little value to the researcher. However, thinking of a research framework in terms of a rationale or outline map (Fulton and Krainovich-Miller, 2010) might be clearer for some researchers. So a research framework viewed in terms of purpose, research questions, literature review, design, data analysis, findings and recommendations may assist researchers in ensuring that their research projects are coherent and achieve the desired outcomes.

From the reading of the literature there appears to be two main avenues which qualitative researchers consider when using a framework to guide their work. The first is to use a framework to define the design of the study (Fulton and Krainovich-Miller, 2010) to create a rich, critical and analytical narrative of a particular problem under investigation. The second is to use a framework for the explicit generation of theories that would make the research findings meaningful and allow for generalisation (Polit and Beck, 2004). This research study adopts the first approach as advocated by Fulton and Krainovich-Miller (2010) to define, guide and organise the study. This decision was taken because of the rich narrative that was anticipated from the classroom setting and the participating teachers. It was not anticipated that generalisations or theory developments could be made from this study. This was due to the research being undertaken in one setting and focussing on one particular aspect of the curriculum.

Abbott (1998) and Witz (1992) have both researched how frameworks can be used in terms of two features - “professional” and “authority”. Professional refers to the boundaries of work ‘owned’ by a profession (Abbott, 1998). This is in contrast to the term authority which refers to the type of authority that a professional has to undertake whilst they are performing their work (Witz, 1992). These two concepts are both used to frame this study and the research questions. They inform and underpin the collection and analysis of the research from the classroom work, semi-structured interviews with participants and document analysis at the study school. The same data may have been collected if different theoretical concepts had informed the framework for the study, or even if no framework had been used at all, but the likelihood is that the data may have been interpreted very differently. There is no judgement being made here as to the worth or legitimacy of the approach, but a simple recognition that alternative interpretations will exist and that these are dependent on the research framework being used.

The use and definition of an underpinning framework was seen to be the truthful way forward when embarking on this PhD study project. The existence of a framework underpinning the study based on professional boundaries and authority aided the mutual intellectual conversations with participant professionals. It also grounded the research in the everyday practices of the classroom to give it the authority it needed.

1.4 Structure of the Thesis

Therefore, this research is grounded in a qualitative case study methodology to understand the influences of lesson design when pupils are learning to divide fractions. The approach was selected so as to focus on a particular issue (the use of pedagogical language) to examine learning and participants perspectives. I deliberately implemented an approach based on equivalent fractions which used pupils' prior knowledge of fractions rather than the more normal invert and multiply algorithmic procedural approach. Key features of cognitive accelerated learning in mathematics (CAME) (Adhami, Johnson and Sayer, 1998; Shayer and Adhami, 2003 ; Shayer and Adhami, 2006a) underpin the pedagogical and theoretical learning approach used for this research. CAME's theoretical approach is applied to the development of the materials but this study extends the approach to the development of mathematical learning sequences with clear pedagogical definitions of the terminology.

Chapter 2 reviews the literature relating to theories of learning and teaching mathematics; fractions pedagogy and the language of lesson design. It is argued in this chapter that mathematical knowledge, pedagogy and the language of lesson design are inextricably linked. From the review of the literature it would seem that the links between the learning of conceptually difficult topics (such as fractions) and lesson design principles are worthy of additional investigations and study.

Chapter 3 describes and justifies the research methodological and design approach for the analysis of the data collected from the classroom study. The aim is to identify elements or themes that apply to the design of lessons, particularly those relating to mathematically conceptually difficult topics. This chapter also includes the details and descriptions of the data instruments and methods used in the study. Once the research questions had been written and a methodological approach selected, I then started to think about the data methods I would need to use to create a rich qualitative description of the context and situation. Also, having examined a considerable amount of literature relating to how ethical research should be conducted, most of my ideas were theoretical in nature. I decided that before I could continue with the main research I needed to test what I had gained from the literature and the various assumptions I was inevitably making. I consequently approached a school, which was the one I was going to use for the research, to set up a pilot session so that I could test a number of ideas and assumptions all at the same time. First, I could test the data tools and the intended data collection methods (including the use of video) to see how pupils would react and how much data would be generated. Second, I could test some of the teaching materials and pedagogical terminology with a small group of pupils and their teacher. Thirdly, I could establish a professional relationship with all those involved and start the conversations about the professional boundaries and authorities of the research (Witz, 1992; Abbott, 1998). So this chapter also has details of the pilot study that was conducted.

The findings (chapters 4 and 5) from the study together with the analysis and discussion (chapter 6) form an extensive part of the whole study. The research led to an analysis of learning of fractions across the attainment range, the relationships between participants and their beliefs about teaching and lesson planning and the impact of a common understanding of lesson design terminology. The analysis of the data from questionnaires was more quantitative and deductive, but the vast majority of the data collected was analysed from an inductive, observational standpoint and as such obviously contains some value judgements both from the researcher and the teacher participants.

The final chapter presents a set of thoughts resulting from the research study and places it within a research context and points to implications and future avenues for research. I hope that this study helps to generate a conversation about pedagogical lesson design terminology and raises questions about the teaching and learning of mathematics for the benefit of teachers, but much more importantly for our children.

2.1 Introduction

I have decided to write this chapter in such a way that it charts my exploration and investigation of the literature. Having framed the two main research questions

> **RQ1. What are the influences of lesson design on pupil learning?**
>
> **RQ2. What are the implications of a change of approach to teaching fractions for teacher training?**

my initial thoughts were to section the literature review into four parts:

1. Learning mathematics (RQ1)
2. Teaching mathematics (RQ2)
3. Lesson design terminology (RQ1)
4. Professional development of mathematics teachers (RQ2).

It soon became apparent that this approach was neither allowing me to synthesise the literature for interconnections between various aspects contained in the research questions, nor to coherently inform my research. So, I decided to search the literature using a different approach and look at key themes contained in the two research questions. Not surprisingly there was a wealth of relevant research concerning the key themes of pupil learning, the teaching of fractions and lesson design. Making sense of this body of research to select and critically evaluate its applicability to the above two research questions is what the remainder of this chapter is about. By using the three key themes (pupil learning, the teaching of fractions and lesson design) the chapter explores the two conjectures, relating to learning and teaching mathematics, that there:

> is a distinction between mathematical knowledge (what should be taught – factual knowledge) and pedagogical knowledge (how the subject should be taught);
>
> and that the pedagogy of mathematics (how the subject should be taught) requires a common, shared conception of lesson design terminology so as to have a positive impact on pupil learning and teachers' shared conversations.

These two conjectures stem from a long career in mathematics teaching where I have become increasingly intrigued as to why the majority of pupils find the subject so difficult to learn (Ruffell, Mason and Allen, 1998; Brown, Brown and Bibby, 2008; Collie et al., 2018).

There are large parts of the population who hold extremely negative views, fears or even a hatred of mathematics as claimed in the literature (Handerson, 1981; Sewell, 1981; Mtetwa and Garofalo, 1989; Frank, 1990; Ernest, 1996; Reinup, 2009; Mirza and Hussain, 2018). People who hold this negative image of mathematics often view the subject content as difficult, impenetrable and abstract (Buxton, 1981; Lewis, 2014). But does this view have a basis in the research? Cockcroft (1982) found most people stopped in the street would not talk about mathematics and this tendency was replicated in an international study of seven countries by the Basic Skills Agency (1997). There is a view that "learning mathematics is a question more of ability than effort" (McLeod, 1992, p. 575) and that some people have “an inherent natural ability for mathematics" (FitzSimons et al., 1996, p. 768; Cai et al., 2018). The myths about mathematics as a difficult subject to learn (Cockcroft, 1982; Perez-Felkner, Nix, and Kirby, 2017), a subject only for the clever ones and a belief that it is predominantly a male subject (Shuard, 1982, 1986; Burton, 1989; Burton 1998) have all been researched and challenged in the literature. A study by Cai et al. (2018), which involved 241 Chinese children, found that their self-conception “of ability exerts a positive effect on math skills even in cultures that emphasize effort over ability” (p. 71). So the question arises as to whether these prejudices result from the fact that mathematics is a difficult subject to learn, or is it from the quality of mathematics teaching or a combination of both of these factors (Nurlu, 2017). This prompted me to explore the literature in more depth for what is known about good and not so good learning in mathematics, the teaching of fractions and lesson design.

2.2 Theoretical perspectives of learning

This section critically reviews the literature relating to learning and specifically the learning of mathematics. In order to explore what is meant by learning mathematics I wanted to know what others working in the field have come to understand about mathematics learning and the supporting learning theories.

Looking at the relevant learning theories and how they apply to mathematics learning would also help me to identify gaps in the literature relating to my research question 1, and what types of learning episodes might support pupils' learning of mathematics.

The consideration of appropriate learning theories would also help to highlight why pupils find mathematics difficult to learn and in particular why some pupils struggle with learning certain topics such as fractions (Davis et al., 1993; Boulet, 1998; Morris, 2001; Hunting, 2003; Thompson and Saldanha, 2003; Clarke, 2006; Woodward, 2017). The learning theories may also assist me in understanding why, as reported in the literature, the learning of fractions is probably one of the most serious obstacles to the mathematical development of children (Behr et al., 1993; Siegler and Pyke, 2013; Illeris, 2018).

Underpinning this study are two interconnected assertions about mathematics learning. Firstly learning is likely to be situated in a context which is both socially experienced and academic in nature, and the context is an important aspect to how a person learns a particular piece of knowledge (Greeno et al., 1996). Cobb and Bowers (1999, p. 5) contextualise this view for mathematics arguing that situated learning theorists often compare "mathematical activity in school with activity in various out-of-school settings". Situated learning might therefore appear to align nicely with the utilitarian perspective of mathematics (Skemp, 1978; Moreno and Rutledge, 2018). However, cognitive psychologists argue that learning is an individual act, where the learner is a processor of information and mathematical symbols (Núñez, Edwards and Matos, 1999). So in contrast to a situated learning theorist's a cognitivist's view of learning might appear to align nicely with the purist's perspective of mathematics (Rowe, 2018). Secondly, a teachers' subject knowledge and beliefs relate to their own educational experiences; the political and social contexts in which they practice to create the situations in which learning occurs (Schoenfeld, 1998). Calderhead and Robson (1991) reported that trainee teachers often held strong images of what learning entails and the enactment of teaching from their experiences as pupils. Witnessed classroom practices form an influential role in determining how trainee teachers translate observed practices for later use when they qualified (Bergman, 2018).

2.2.1 Overview

The historical, but long standing, view of the dominant mathematical learning psychology was derived from a Euclidean approach mediated through Socratic dialogue (Lerman and Cowley, 2012). There was a requirement to accumulate knowledge and facts that were acquired by an individual in response to a direct stimulus, and hence learning was viewed in terms of a behavioural response (Illeris, 2018). As a counterargument to the passive behaviourist learning approach, a constructivist would argue that learning is an active process of contextualising the creation of knowledge rather than its mere acquisition. Constructivism therefore argues that new knowledge is formed and built from the learner's previous knowledge and this is independent of how the knowledge is presented. A constructionist would advocate that knowledge is created and based on personal experiences (Lerman and Cowley, 2012). Learners are continuously forming and reforming their conceptions of knowledge through social interactions and negotiations, indicating that learners have differing interpretations and constructions of knowledge (Ertmer and Newby, 1993).

A social constructivist would view learning as an active process where pupils should learn to discover principles, concepts and facts for themselves with the aim of encouraging intuitive thinking in learners (Brown, Collins and Duguid, 1989; Ackerman, 1996; Lerman, 1999). Kukla (2013) argues that reality is constructed by our own activities and that people, working together as members of a society, invent the properties of the world. Constructivists generally agree with this and emphasize that individuals make meanings through the interactions with each other and with the environment. Knowledge is therefore a product of human interactions and is socially and culturally constructed (Ernest, 1991; Prawat and Floden, 1994; McMahon, 1997). Vygotsky (1980) also highlighted the merging of the social and practical elements in learning. His view is that the most significant moment in the course of intellectual development occurs when speech and practical activity, two previously completely independent lines of development, converge. Through practical approaches to learning a child constructs meaning on an intrapersonal level, while speech connects this meaning with the interpersonal world which is shared by the child with others and the interaction with their mutual culture.

Social constructivist theory therefore explains the process by which a learner gives meaning to knowledge, which is actively experienced and is a “meaning-making philosophy that informs pedagogical practices [that has] dominated the past several decades of educational practice” (Schrader, 2015, p. 23). Furthermore any one person’s understanding or construction of knowledge is as ‘true’ as any other person’s implying that all interpretations that ‘work’ are equally valid and that no single ‘truth’ exists (Dickerson and Zimmerman, 1996; Doan, 1997). Complementing Doan’s (1997) view of social constructivism Dickerson and Zimmerman (1996, p. 80) argue that learning “locates meaning in an understanding of how ideas and attitudes are developed over time within a social, community context”. Whereas Hoffman (1991, p. 5) states that all “knowledge evolves in the space between people, in the realm of the ‘common world’”.

Social interaction plays an essential part in the process of cognitive development (Vygotsky, 1980; Lerman and Cowley, 2012; Light, 2017) and contrasts with Piaget’s (1952) four-stage model of cognitive development where a learner progresses through all four stages in the same predetermined order. Vygotsky felt social learning precedes cognitive development, where “every function in the child’s cultural development appears twice: first, on the social level, and later, on the individual level; first, between people (interpsychological) and then inside the child (intrapsychological)” (Vygotsky, 1980, p. 57).

In comparison to the social constructivists theory of learning the 1960s view was that of a cognitive process where the learner passively received and processed information. Learning was viewed as an internal activity and separate from external influences with learning resulting from an individual’s ability to observe, organise and process the information they were being given. Piaget (1936) identified a four stage model of child development (Sensory-motor, Pre-operational, Concrete operational and Formal operational) which intimately linked with the learning process. Critics including Bruner and Vygostky cited a number of flaws in the model such as learning should not be viewed in distinct, separate stages but more as a continuous process, and that Piaget’s theory underestimated a child’s abilities at particular cognitive developmental stages. This underestimation was a direct result of Piaget’s observations being undertaken as a single researcher and his results, findings and interpretations are consequently more open to bias, subjectivity and challenge (Goos, Galbraith, and Renshaw, 2004). Even though Piaget's findings have been questioned over the years they "have been shown to be robust" (Fasko, 2016, p. 2).

It can therefore be argued that there are flaws in Piaget's research design methodology and this calls his model to be challenged. However wider problems exist with Piaget's theory of learning such as language only being considered as a secondary feature (Piaget, 1959). This omission has helped to further discredit Piaget's theory as argued by Vygostky (1978) who considered language as the most important medium to communicate reasoning and thought and consequently facilitate cognitive development.

Bloom's (1959) six stage hierarchical cognitive process pyramid (Knowledge, Comprehension, Application, Analysis, Synthesis, Evaluation) is often quoted and used in classrooms as a means of developing learning and thinking (Chikiwa and Schäfer, 2018). This notion of learning being partitioned into separate tiered entities is again, by the same reasoning, flawed and not supported by any large body of research. However research does support the distinction between declarative knowledge (knowledge recall and comprehension) and procedural knowledge (application of knowledge to a task), but these are interdependent and not separate as Bloom's cognitive process pyramid might imply.

Bruner focused on conceptual understanding, cognitive skills and learning strategies rather than the acquisition of declarative or procedural knowledge, Bruner (1964) theorised that human intellect developed "from infancy to such perfection as it may reach [and] is shaped by a series of technological advances in the use of mind" (p. 1) and these advances are dependent on increasing language development, the mastery of techniques and systematic teaching. Bruner (1996) criticised the way in which his view of the developmental learning sequence (enactive, iconic and symbolic) was perceived by others because of their implications of categorising the stages in learning: he preferred to view the sequence (enactive, iconic and symbolic) as modes of learning (Hevern, 2003).

Ausubel (1968) believed in meaningful, as opposed to rote learning and stressed the importance of engagement in active mental participative learning tasks. He viewed the single most important factor influencing learning to be what the individual already knew (Ausubel, 1968; Hansen, 2009; Kambouri, 2016) and that learning was the direct result of being exposed to new knowledge rather than self-discovery. The need to expose learners to new knowledge resulted in his view of learning being facilitated by two types of advanced organisers (comparative and expository). Comparative organisers activate learning and they act as reminders to bring into memory what is relevant: in contrast expository organisers often relate what the learner already knows with the new and unfamiliar material that is to be learnt. Ausubel recognised the criticisms of his theory arguing that

> The most persuasively voiced criticism of advance organizers is that their definition and construction are vague and, therefore, different researchers have varying concepts of what an organizer is (Ausubel, 1978, p. 251).

However, it was the Vygotskian social constructivism revolution of the 1970s and 1980s that started to change the view of learning from that of a purely cognitive process. Vygotsky's views stressed the significance which social and linguistic influences have on learning and in particular on the role of the teacher. He introduced a term, 'the zone of proximal development' (ZPD), to describe the difference between what a learner can do without help and what can do with help. In Vygotsky's own words, the ZPD is "the distance between the actual developmental levels as determined by independent problem solving and the level of potential development as determined through problem solving" (Vygotsky, 1978, p. 86

The implication from the above is that there may be potential for a pupil to reach much higher conceptual levels of understanding, given appropriate teaching, than that would be normally expected. Vygotsky went further theorising that ZPD "enables us to pro-pound a new formula, namely that the only 'good learning is that which is in advance of development '" (Vygotsky, 1978, p. 89).
The move from the individualistic cognitive formation of knowledge towards a socially constructed viewpoint is being currently challenged by society and policymakers. Research (Tosto et al., 2016) is suggesting that there is a much greater emphasis on individualised learning for examination performance, and hence there appears to be a return to the cognitive acquisition of knowledge as advocated in teaching approaches such as the Secondary Mathematics Individualise Learning Experiment (SMILE) of the 1970s and 1980s. The Vygotskian social constructivist view of learning is being challenged by approaches geared towards examination performance and is increasingly giving

way to learning appearing to be moving back towards a version of Piagetian cognitive acquisition of knowledge (Moon, 2009). Nevertheless Brooks and Brooks (1993) remind us that

> Constructivism is not a theory about teaching…it is a theory about knowledge and learning… the theory defines knowledge as temporary, developmental, socially and culturally mediated, and thus, non-objective (p. vii).

Naylor and Keogh (1999, p. 93) add that “learning involves an active process in which learners construct meaning by linking new ideas with their existing knowledge”. This constructivist theory is in direct comparison to the individualised learning that appears to be becoming more prevalent.

The tension between the views of Driver et al. (1994) relating to mathematical knowledge being socially constructed and the view that all knowledge is individually constructed as a result of purposeful activity, might be considered to be at the two ends of a continuum. The union of these two points of view was the theme of the 1992 conference on alternative constructivist epistemologies (Steffe and Gale, 1995); where they state the “intention is to establish possible relationships amongst alternative [constructivist] epistemologies .... [and that] there is a lot at stake here for the education of children” (p. 489).

Lave (1988) argues that learning normally occurs as a function of an activity, context and culture in which it is situated and is frequently unintentional rather than being deliberately planned. This might be considered to be in complete opposite to classroom learning activities (appendix 50 – problem B - Ratio) where the mathematics is often abstract and out of context, yet planned. There is a distinction here in that frequently mathematics learning activities, which involve acquisition of knowledge, are often abstract and out of context. With social interaction being a key component of situated learning, learners become involved in a "community of practice" which embraces certain beliefs and behaviours and is based in real contexts. With situated learning tending to have characteristics of problem-based learning and problem solving (which are key components of mathematics education) and require a degree of learner independence. The argument is therefore that mathematical problems are often rooted in a particular context or subject domain (a situation).

A significant feature of situated learning, as a general theory of knowledge acquisition, is that of social interaction. Social interactions are not always a common, observable feature in mathematics learning or classrooms. The lack of this type of learning can often result in dissatisfaction or tedium if learners are required to work in isolation and silence (Nardi and Stewart, 2003, p. 345). Brown, Collins and Duguid (1989) add to the theory of situated learning by proposing the notion of cognitive apprenticeship to support:

> learning in a domain by enabling students to acquire, develop and use cognitive tools in authentic domain activity. Learning, both outside and inside school, advances through collaborative social interaction and the social construction of knowledge (p. 34).

The notion of learning from the interactions with others and from their experiences is therefore a central feature of situated learning. What initially appear to be two quite separate and distinct theoretical approaches to learning can be viewed from the standpoint of complementary theories which can enrich the learning experience. As noted by Núñez, Edwards and Matos (1999, p. 47) "the situated learning perspective was welcomed by educational researchers and theorists as a richer and more appropriate means of addressing cognition". Perera (2011) therefore argues that situated learning and constructivist theories could be viewed as compatible and appear to be mutually supportive.

This complementary stance of two learning theories led to a view of situated cognition *(*Brown, Collins and Duguid, 1989) where learning and doing mathematics are bound together in *s*ocial, cultural (real-world problems) and physical contexts. Situated learning therefore requires a modification of an epistemological position from learning being a solitary experience towards a model where learning is dynamic, active and less reliant on passively receiving, storing and retrieving knowledge.

When learning is viewed as a social action it is facilitated through interactions with the environment and other individuals which provide opportunities for members of the community to learn through relationships with their peers. Kapucu (2012) demonstrated that classrooms can be a community of practice and that activities facilitate learning. Pupils are given the opportunity to become rooted in a social learning environment with their peers and teachers who act as "facilitators of the classroom community, and building a community of practice"

(Kapucu, 2012, p. 606). For classroom communities of practice to be effective teachers need to "use appropriate strategies to promote collaborative activities within the classroom" (Kapucu, 2012, p. 606).

So, taking this viewpoint that learning is situated, the contexts in which the individual is placed impacts on the knowledge acquisition, skills and understanding which the learner gains (Greeno et al., 1996). Lave (1988) implies a strong connection between a situated viewpoint of learning and the social context and that this associated relationships is a key parameter for learning. Cobb and Bowers (1999) researching mathematical learning disagree with this description, instead they argue that a situated learning standpoint offers a variety of viewpoints "the choice in any particular case being a pragmatic one that depends on the purposes at hand" (p. 6).

The theory of 'situative learning' or the situations in which learning takes place as a social activity (Lave, 1998) both contrasts and complements the cognitive stance which focuses on the acquisition of knowledge. Moon (2009, p. 11) describes learning as "a process [which is] in constant flux and is difficult to describe in a linear manner" but a variation or a variety in the situation prompts learning when "there is a change in the external experience in the material learning" and "the learner works with her internal experience and relates it to other prior experience" (Moon, 2009, p. 28).

The main concept behind Lave and Wenger's (1991) situated learning is "the desire to explore how people learned new knowledge and skills without being part of formal training" (Aubrey and Riley, 2019, p. 212). This notion embodies the concept that learners

> involve themselves in communities with other practitioners to develop their practice. Learners become increasing adept at their mastery of skills and knowledge which they gain from more experienced practitioners, eventually developing into fully fledged members of the community (Aubrey and Riley, 2019, p. 212).

This prompted my thinking about classrooms as communities of practice where pupils learn from each other and from the teacher, with the mastery of skills and knowledge perhaps leading them to consider joining the mathematics community. Additionally the idea of new knowledge not needing to be part of formal training was intriguing and further prompted my thinking about alternate ways of promoting pupil learning. In dynamic interactive community of practice where skills and knowledge are socially developed:

> the individual learner is not gaining a discrete body of abstract knowledge which (s)he will then transport and reapply in later contexts. Instead, (s)he acquired the skills to perform by actually engaging in the process (Lave and Wenger, 1991, p.14).

Applying a ‘community of practice’ to the classroom was seen as problematic by Hooks (2003) because of practices such as assessment and delivering an accredited curriculum. Avis et al. (2010) counters the argument by offering the advice that pupils should value assessment as part of the learning process and not as a final outcome. The notion of a community of practice embedded in the classroom still held a level of fascination and I felt it needed to be explored as part of this research study.

The theories of learning are hence pointing towards the benefits of learners collaborating through the medium of language with peers, teachers, and other people to develop individual understanding of knowledge (Dougiamas, 1998; Wolfe, 2010). Jaworski (1996, 2007) connects learning to a community of practice as defined by Lave and Wenger (1998) in their Social Practice Theory (SPT). Jaworski (1996) summarises this connection to a community of practice as:

1. knowing is an action participated in by the learner. Knowledge is not received from an external source.
2. Learning is a process of comparing new experience with knowledge constructed from previous experience, resulting in the reinforcing or adaptation of that knowledge.
3. Social interactions within the learning environment are an essential part of this experience and contribute fundamentally to individual knowledge construction.
4. Shared meanings develop through negotiation in the learning environment, leading to the development of common or 'taken-as-shared” knowledge.
5. Learning takes place within some socio-cultural setting – a ‘community of practice’ in which we can think of social actions as well as social interactions (Jaworski, 1996, p. 6).

In this section I have outlined what initially appears to be a number of distinct and separate learning theories and have argued that they complement each other. Bridging the divide between the view of learning as a social, constructivist activity and the views of cognitive learning theory is of real and arguably influential importance in the learning of mathematics (Ernest, 1991). With language as a central feature of shared understandings in a community of practice the evidence for learning based on socially constructed paradigms is compelling. However as Ernest (1991) warns there is an important feature shared by all learning theories that of an imperfect view of knowledge and of mathematical knowledge in particular. With Ernest's (1991) stark warning in mind I next needed to research how these theories might apply to mathematics and more particularly what practical conceptions of the theories are applicable to the learning and teaching of mathematics.

2.2.2 How do children learn mathematics?

Before considering the ways in which pupils learn mathematics we ought to briefly first consider the nature of mathematics as a discipline. A way of defining mathematics could be by the types of problems it deals with and the methods, procedures and algorithms used to solve these problems. The perception of the nature of the subject often lies on a continuum from the utilitarian where the subject deals with solving real world problems to the purely esoteric beauty of constructing ever increasing layers of mathematical knowledge (Ernest, 2013). There is some compelling evidence about the nature of mathematics that suggests the experiences of learning the subject from either of these two perspectives (utilitarian and purist) is connected and interdependent (Thompson, 1984; Lerman, 1990; McNamara et al., 2003; Ernest, 2016).

The didactic approach to learning was prevalent in mathematics education of the past where learners were required to individually work on exercises and problems from sources often hundreds of years old. This view of learning mathematics was dominated by mechanical drill exercises where learners were required to produce identical solutions using the given or prescribed method. The main justification for this approach to the learning of mathematics, and to some extent it still is today, was that even the dullest and most uninspiring approach develops a child's mental ability. By the 1970s and 1980s the didactic view of learning mathematics was being replaced by a vision of learning as an active, social process where learners construct or create their own representations of knowledge. The work of Vygotsky (1978) on the themes of learning through social interaction, the more knowledgeable other and the zone of proximal development were, as discussed above, in direct contrast to Piaget's (1952,

1976) view of a child's development of the mind as a necessary precursor to learning.

The Piagetian view of education as a means of supporting a child's needs and interests is in direct contrast, but arguably potentially complementary, to that of the societal transformation view rooted in Vygotsky's social constructivism. The very nature of mathematics and the ways in which new mathematics knowledge is created in the minds of learners is deeply rooted in the constructivist approaches to learning (Schifter and Simon, 1992). To achieve a depth of understanding of mathematics pupils "must be actively engaged in reconstructing their existing understanding by reconstructing their cognitive maps" (Richardson, 1997, p. 5) as opposed to simply acquiring and passively receiving knowledge for factual recall. Constructivist learning theory can therefore be viewed in terms of a social activity with an individual pupil creating their own understandings of mathematics through the interactions with others and obviously this does not take place in a vacuum. So the argument is that mathematical learning has to be achieved through experience and social interactions where

> It is assumed that learners have to construct their own knowledge-- individually and collectively. The role of the community-- other learners and teacher-- is to provide the setting, pose the challenges, and offer the support (Davis, Maher and Noddings, 1990, p. 3).

Pais (2013, 2017) makes the point that the value of school mathematics lies more in its perceived usefulness within society than the inherent beauty of the subject. Pais (2013, 2017) disagrees with Ernest (2000, no page) suggesting that "the utility of academic and school mathematics in the modern world is greatly overestimated". At the other extreme mathematics can be "a source of delight and wonder" (Ollerton, 2006, p. 19), and a subject which is "beautiful, intriguing and mind blowing" (Ollerton, 2006, p. 52). It is "a major intellectual discipline in its own right" (Smith, 2004, p. 2) that can "stimulate moments of pleasure and wonder for pupils" (QCA, 2007). Radford (2003) argues that "the humanist view of mathematics emphasizes the role this discipline plays in the development of logical thinking, abstraction, rigor and other highly prized faculties ... and is seen in terms of the utilitarian"(p. 552) nature for mankind.

Naidoo and Parker (2005), Darragh (2016) and Sjölund (2018) all identify a centralist view of mathematics on the continuum from utilitarian to purist, arguing that the subject is a force for social change. Ernest (1991) takes the complete opposite view arguing that mathematics is a rigorous system of pure timeless truth, universally valid and both value and culture-free. Nevertheless at the heart of this continuum is an interdisciplinary mathematical language as a tool for problem solving (Usiskin, 1996). It is this tool that is considered to be the basis of our formal mathematics education system (Thompson and Rubenstein, 2000; Armstrong, Ming and Helf, 2018). Many pupils and their teachers are therefore predisposed to identify the mathematics that they learn in terms of this interdisciplinary language and this is often geared towards final public examination assessments rather than any one view of the nature or utility of mathematics.

2.2.3 Learning approaches in mathematics

Critics argue that Piaget's work is not a complete description of cognitive development (Eggen and Kauchak, 2003). Gelman, Meck and Merkin (1986) conjectured that Piagetian theory underestimates the abilities of young children; whereas Eggen and Kauchak (2003) criticise Piaget for overestimating the abilities of older learners. The notion of the possibility of explicitly being able to teach using a cognitive development approach was investigated by Adhami, Johnson and Shayer (1997). The research from the two projects Cognitive Acceleration in Mathematics Education (CAME) and Cognitive Acceleration in Science Education (CASE) demonstrated that the approach is able to accelerate pupil learning (Adhami, Johnson and Shayer, 1997; Shayer, 1999). Piaget believed that not all pupils in a class are operating at the same cognitive development stage; therefore it might be a strong argument for self-differentiating, open-ended tasks (appendix 50) as a way of enhancing a pupil's learning of mathematics.

An alternative approach to the learning of mathematics might therefore be to have learners in mixed aged groups where pupils are at different cognitive developmental stages. There is a reasonable body of supporting research that support this view, albeit in small schools (Cornall, 1986; Galton and Patrick, 1990; Francis, 1992; Vulliamy and Webb, 1995). However, research by Veenman, Lem and Roelofs (1989) concluded that there is no significant impact on pupil cognitive development when taught in multi-aged classes. Teachers working in mixed-aged groups were often teaching as if they were two or more different classes rather than considering all pupils to be at the same cognitive developmental stage.

Most abstract mathematics begins in secondary school and this is not solely confined to just algebra. Topics containing abstract ideas such as fractions, geometry and probability all have their beginnings in the early secondary school curriculum. New knowledge, according to Piaget, is built up through experiences which are then checked and validated against existing knowledge. He argues that new knowledge has to be assimilated and existing concept structures have to be reorganised or modified. Immature pre-operational thinkers can learn procedures or algorithms but do not develop conceptual understanding of abstract ideas. According to Burns and Silbey (2000, p. 55) “hands-on experiences and multiple ways of representing a mathematical solution can be ways of fostering the development of this cognitive stage”.

The child at the early secondary school (ages 11 to 14) is operating in Piaget’s formal operations stage (mainly due to the ways in which they are taught) and should be capable of structuring concepts as a foundation for the development of more abstract thought patterns where reasoning is executed using symbols without the need to rely on manipulative materials. At this developmental stage the learner can solve $2x + 5x = 14$ without having to refer to a concrete representation. Contrastingly at the next cognitive developmental stage learners are capable of forming hypotheses and deducing possible consequences, to enable the child to construct their own mathematics.

This has implications in that the majority of children have developed a schema, or learning map, for the four mathematical operations on integers during the concrete developmental stage (mainly during the ages 7 to 11 – Primary School Curriculum). Unfortunately pupils might then fail to apply these operation schemas correctly to the rational numbers when trying to develop their own view of the next steps in learning mathematics (Tall, 2002). The extension of the four operations over the rational numbers is often begun with manipulative materials during the formal operations stage. The cognitive development of these four operations on fractions very quickly moves on to a much more abstract approach being taken by teachers. This is usually by considering the required, or formally recognised, algorithms and procedures. The problem is then that pupils are often not able to link what has been previously learned with the abstractness of the procedures and algorithms. Research by Gabriel et al. (2013) who studied 21 mathematics textbooks, interviewed 24 teachers and analysed the results of 439 test scripts for 9 -11 year olds relating to fractions, found that the practice of focussing on procedures is not sufficient because “conceptual understanding is essential to ensure a deep understanding of fractions”(Gabriel et al., 2013, p. 9).

The Vygotskian view of the social formation of the mind through scaffolded talk as a means of promoting understanding and reasoning has its basis in situated learning. Vygotsky's theories embody social interaction as a fundamental element in the development of cognition where "all the higher functions originate as actual relationships between individuals" (Vygotsky, 1978, p. 57). With situated learning embedded in the theories of the social learning and problem solving of Schoenfeld (1985) a general theory of knowledge acquisition can be seen to apply to the learning activities that focus on problem-solving skills. Eisenhart and Borko (1991) express meaningful learning in the form of active knowledge construction where learning occurs “as they [pupils] modify and elaborate their knowledge structures through a process of adaptation to the environment” (p. 142).

The notion of a learning schema affords theorists from differing epistemological standpoints a common language or model to describe cognitive constructions of learning. The term “schema” as applied to learning can be traced back to a study of memory by Bartlett (1932) which was then developed by Oldfield and Zangwill (1942a, 1942b, 1943) and Skemp (1962, 1971, 1979). Minsky (1975) introduced the notion of “frames” and this was developed by Tannen (1993) as a methodology for the analysis of discourse. Schank (1975) had previously developed the idea of “scripts” as a way of describing conceptual schemas. Both “frames” and “scripts” are closely linked and broadly similar to Bartlett’s schemas. However, whilst the term schema has been applied to mathematics education (Steffe, 1983, 1988; Davis, 1984; Dubinsky, 1992; Cottrill et al., 1996) there have been few attempts to define precisely what might constitute a schema in respect to all or even the majority of mathematical topics. Therefore, rather than viewing mathematics, and in particular operations on fractions, as a set of procedures or facts linked together by algorithms that need to be acquired, a constructivist’s approach to the learning would be to create coordinated mental schemas (Kieren, 1990; Kieren, 1994; Steffe, 1990). By considering the mathematical patterns and associated interlinked relationships learners might be able to self-construct these mental schemas.

Schematic learning when applied to mathematics education introduces the notion that new learning is assimilated, organised and interpreted with reference to past or prior learning (Skemp, 1979, 1986). It is therefore an important pedagogical task for the teacher to discover the schemas a child has internalised and that they are using. Being able to define new learning that builds naturally on the schemas that a pupil is already comfortable with and confidently using is at the heart of teaching. If a pupil has acquired or internalised a schema for division of integers then it would appear that Skemp is arguing that the learning of division of rational numbers should extend and build on the currently held schema, rather than deviating or introducing new schemas for specific cases. The introduction of a new schema for the division of fractions is often the case and this can result in confusion and poor recall. Building on prior schemas relies on the teacher having both solid subject knowledge and pedagogical insights into the pupils' strengths, understanding and the ways they view and interact with mathematics.

If we view learning through the mediation and extension of schemas together with social interactions, then the learning of mathematics through collaboration has been shown to reduce peer competition, promote achievement and foster positive relationships. Research by Swan (2006) into the development of resources, using a model of collaborative discussion to reshape students' existing knowledge when working towards public examinations, indicated that student-centred learning resulted in the greatest gains. According to the National Council of Teachers of Mathematics (NCTM, 1991) Teaching Standard 8, learning should promote active learning and teaching; classroom discourse; and individual, small-group, and whole-group learning. Collaborative learning through classroom discussion can be the stimulus for active learning as noted by Swan (2006, p. 227) "discussion – based approach to learning is to encourage students to move from 'passive' learning strategies to more 'active' ones". Ofsted (2009, 2012) noted that

> pupils become confident learners as they develop skills in articulating their thinking about mathematics. They learn to make sense of ideas, and

> reason and justify their methods and solutions because discussion is a regular feature. Learning is therefore active and cumulative (Ofsted, 2009, p. 12).

> They [the pupils] frequently told inspectors that in other subjects they enjoyed regular collaboration on tasks in pairs or groups and discussion of their ideas (Ofsted, 2012, p. 19).

Ingram, Sammons and Lindorff (2018) whilst reviewing the literature on effective mathematics teaching found that learning mathematics is frequently described in terms of a collaborative activity. Mercer (2000) found that tasks which promote active involvement and encouraged critical, collaborative constructive discussion to be more effective than an uncritical acceptance of rules procedures, algorithms and methods. Swan (2006, p. 162) describes the distinction between a transmission model and the active model of learning as “an individual activity based on watching, listening and imitating until fluency is attained”. The transmission model of learning is in complete contrast to a collaborative learning approach where “learners are challenged and arrive at understanding through discussion”. Yet this transmission model is often seen in mathematics classrooms and considered to be the definitive approach to teaching mathematics. As Watson (2019, p. 2) observes “in-the-moment decisions can lead to a lesson becoming more traditional and teacher-centred even though the teacher may have the knowledge of and hold beliefs in reform-oriented student-centred approaches”.

The structure of a mathematics lesson is often “rigid, characterized by rote learning and endless repetition of mechanical tasks” (Evans, 1994, p. 2). The use of practical elements, entailing “learning by doing” with “the additional feature of reflection upon both action and the result of action” is the key for experiential learning to take place (Capel, Leask and Turner, 2001, p. 252). Ellis (2007) suggests that practical and experiential learning are beneficial to all and are an integral part of the learning process, with Beard and Wilson (2006, p. 18) reminding us that “experiencing something is a linking process between action and thought”; by including experiential activities in the classroom, learners are encouraged to make logical links between theoretical models and real-life practice.

Moon (2009), a researcher in experiential learning, suggests that learning from experience and situations often occurs independently of a teaching process. She also reminds us that the literature on experiential learning (learning from experience) is diverse in nature with no common consensus as to meaning, and each author developing a variant to suit the context in which they are researching. Usher and Edwards (1994, p. 201) argue that when quoting experiential learning as a learning theory “different groups give it their own meanings and construct it in their own ways”. For example, Usher and Soloman (1999) see learning from experiences as being placed in everyday contexts of the real world and is therefore always situated.

If experiential learning implies learning by doing, and this approach requires pupils to actively participate in the learning, then just being a passive learner might not suffice. Boydell (1976) takes the view that “learning is a dynamic, active process, so the trainee learns best by participation. If only a man’s ears are involved (e.g. lecture), much less is learned than if his eyes, muscles, thinking processes and feelings are involved”. Kolb (2015) argues that experiential learning theory describes how real life experiences play an important role in the acquisition of new knowledge. Contrastingly Boaler (2009) argues that “students need to be actively involved in their learning as well as needing to be engaged in a broad form of mathematics, using and applying methods, representing and communicating ideas” (p. 76) implying mathematics is much more than real-life problems.

Brumbaugh and Rock (2013), mathematics educationalists and authors of Teaching Secondary Mathematics, explain that learning by doing or discovery is “a method of indirect instruction where the teacher organises the learning environment, enabling the learner to develop conclusions” (p. 202). Weegar and Pacis (2012, p. 7) support Brumbaugh and Rock suggesting that where learning employs constructivist strategies “students learn by discovering on their own, to students collaborating with others”. Henson (2013) provides us with a working definition for learning by discovery as “intentional learning through problem-solving and under the supervision of the teacher” (p. 101). The use of practical equipment such as manipulatives gives pupils a sense of semi-autonomy or what Brumbaugh and Rock describe as guided discovery leading “the learner in a particular direction toward a desired conclusion” (Brumbaugh and Rock, 2013, p. 144). Guided discovery as a vehicle for pupil learning allows the freedoms and legitimises discussion as well as the construction of a shared understanding of the mathematics.

Mathematics teaching is constantly changing, albeit slowly, in light of developing learning theories and technological advances. Consequently there is no longer a fundamental need to equip learners with prescriptive methods, procedures and algorithms as we are less likely to think of learning defined in terms of a behavioural change or the acquisition of knowledge but more in terms of a social activity (Jarvis, Holford and Griffin, 2003). Hiebert and Grouws (2007, p. 373) argue that "within mathematics, theories of learning have been more clearly articulated than theories of teaching. Although theories of learning provide some guidance for research on teaching, they do not translate directly into theories of teaching". They go on to put the viewpoint that theories which "specify the ways in which the key components of teaching fit together to form an interactive, dynamic system for achieving particular learning goals have not been sufficiently developed (ibid, p. 373). The argument here is that there is interdependence between learning theory and the content specification of teaching sequences which are used to support the learning theories. The implication being as learning theories are constantly developing then the associated theories of teaching need to adapt and the approaches to teaching mathematics need to change.

Having examined the relevant learning theories the core constructivist principles that are effective for mathematics learning can be summarised as being

> Learning mathematics is active and often situated in a context
> Learning involves prior knowledge, experience and discovery
> Learning requires social interactions mediated through language

However, whether pupils are taught using a cognitive or constructivist approach utilising real life problems through guided discovery or by the more traditional didactic approach the resulting impact on the learner would have profound effects for their view of mathematics. As Hoyles (1982) found, secondary school pupils tend to associate their mathematical experiences with feelings of anxiety, shame, and a sense of failure and this is often linked to the teaching approaches taken to engage learners. This theme is explored in the next section.

The literature has led me to the belief that the learning of mathematics is a complex process and more than an individual cognitive process (Piaget, 1952).

Learning is more likely to occur if lessons are designed and based around real-life situations with pupils and teachers working in what Lave and Wenger (1998) call a community of practice. Furthermore knowledge is constructed through social interactions and is mediated through language and discussion and that these should be features of a lesson design. I therefore decided to test these theories by designing, for this research study, a lesson using these concepts where pupils were learning the division of fractions.

2.3 What is known about teaching in mathematics?

It has long been recognised that teaching is a complex, multi-faceted activity (Shulman, 1987; Hill, Ball and Schilling, 2008) and as such, when selecting which characteristics to focus on to define the act of teaching, it is almost impossible to do justice to the depth and breadth that a limited selection might have in classroom interactions. Bowe and Gore (2017, p. 353) state that "figuring out what constitutes good teaching at the local level can take up so much time as to be counterproductive and frustrating". However, Remillard (1999) reminds us that teachers approach classroom interactions in a variety of different ways, but many studies seek to explain teaching characteristics in terms of teachers' beliefs (Thompson, 1992; Wilson, Rozelle and Mikeska, 2011; Bowe and Gore, 2017) and subject and pedagogical knowledge (Fennema and Franke, 1992; Ball, Thames and Phelps, 2008; Cobb and Jackson, 2011; Charalambous, Hill and Mitchell, 2012).

At the outset of this research I therefore wanted to know if the typical didactic, transmission style of secondary school mathematics lessons, which I had almost universally witnessed during my career, had a basis in teacher beliefs, subject and or pedagogical knowledge. I was also interested to find out if the transmission style of delivery resulted from lessons based on a particular planning format or delivery style (such as the explanation of worked examples) from the teacher followed by pupils working through an exercise of similar repetitive problems. This impression of what might be considered to be a fairly standard mathematics lesson format is what anecdotally I had often heard when talking with pupils, teachers and trainee teachers and not surprisingly it does have some basis in the literature as reported by Goos, Galbraith and Renshaw (2004, p. 37)

> In the majority of contemporary classrooms, learning mathematics is seen as mastering a predetermined body of knowledge and procedures. The teacher’s job involves presenting the subject matter in small, easily manageable pieces and demonstrating the correct procedures or algorithms, after which students work individually on practice questions.

This approach to the teaching of mathematics can leave students with imperfect understandings and flawed beliefs about the subject. Cobb (1986) and Cob and Bausenfeld (1995) reported this when students’ are limited to imitating the technique prescribed by the teacher, they can create the appearance of mathematical competence by simply memorising and reproducing the correct way to manipulate symbols, and may be even come to the believe that producing the correct answer is more important than making sense of what they are doing.

After reading a number of mathematics education texts and research articles I did find additional evidence that suggested the above teaching approach was prevalent but it was also contrasted by a body of evidence about alternate approaches and pedagogies. Taking a constructivist approach “the teacher's role is that of a facilitator and requires considerable· reflection as the teacher must observe student responses, challenge student thinking and encourage risk taking within a supportive classroom environment” (Anderson, 1996, p. 31).

I openly recognize that what is known about teaching, and teaching mathematics in particular, is a wide and diverse field of expertise. Given the reported shortage of teachers of mathematics (NAO, 2016; Dickens, 2016) and the general populous view of mathematics, I wanted to focus on a fairly narrow set of characteristics. I therefore, started thinking about how mathematics teachers view themselves as mathematicians and whether their self-confidence with the subject content was a factor that might affect their teaching performance or styles of delivery and pedagogy (Ekstam et al., 2018). I also wondered if the teaching approaches they adopt were dependent on the self-view of their own mathematical attainment. I speculated that the two factors of confidence and performance were probably interrelated. The confidence with the subject content possibly influences them as teachers as they risk losing the confidence of their pupils. Or if they are only one page ahead of their pupils with the subject matter then this might be influencing their pedagogical decisions. Teachers might be adopting a particular teaching approach to mask their lack of subject content confidence such as being overly authoritarian or didactic in order to close down the opportunities for pupils to deviate and ask difficult questions.

2.3.1 Teacher Self-Efficacy

Researching the literature, self-confidence is often closely aligned with self – efficacy and is described in terms of: general teaching efficacy, mathematics teaching efficacy, and mathematical topic teaching efficacy (e.g. fractions) (Riggs and Enochs, 1990; Pajares, 1996; Tschannen-Moran, Hoy and Hoy, 1998). In a study across 16 schools involving 571 pupils Sarac and Aslan-Tutak (2017, p. 66) demonstrated that “students of teachers who had high trigonometry teaching efficacy got higher scores on the trigonometry self-efficacy scale, than students of teachers with low trigonometry teaching efficacy”. Importantly their findings demonstrated that teacher self-efficacy and pupil self-efficacy were linked “since students’ self-efficacy and teacher efficacy are crucial for students’ motivation and achievement, it will be beneficial for educators to understand their relationship” (Sarac and Aslan-Tutak, 2017, p. 67). It would therefore appear that self-efficacy has an influence on motivation and that this has an important function in mathematics education. Confident teaching based on self-efficacy resulting from good subject knowledge (Wright, 1988) which results in pupil interest, motivation and enthusiasm for the subject has a positive impact on both academic achievement and pupil self-efficacy (Pajares, 1996).

Efficacy concerning general teaching practices has been described by researchers such as Tschannen-Moran, Hoy and Hoy (1998) to be the ways in which teachers cognitively assess their confidence with their teaching performance and how they then view their personal teaching competence. For Ross (1992, p. 51) “teacher efficacy measures the extent to which teachers believe their efforts will have a positive effect on student achievement”. Ross (1992) was able to demonstrate with 18 trainee teachers in 36 classrooms that supportive coaching to select appropriate teaching practices resulted in increased teacher confidence which contributed to a teachers’ efficacy, hence demonstrating a link between confidence and efficacy. However, the study was unable to show that “coaching and teacher efficacy would interact such that high-efficacy teachers would benefit more from coaching than low-efficacy teachers" (Ross, 1992, p. 53) instead I would venture that more importantly all teachers' efficacy improved as a result of the coaching.

Much of the recent research related to self-efficacy (a self-awareness about whether one can achieve a specific task or not) is based on the social cognitive theory of Bandura (1982), which states that learners make their choices according to their self-knowledge. Teachers can help pupils to develop self-efficacy through "the use of influential teaching methods" (Sarac and Aslan-Tutak, 2017, p. 66) but the authors do not elaborate on what these methods might be. However, studies have found that when teaching strategies are related to teacher efficacy (Ashton, Webb and Doda, 1983) then the teaching might have positive influences on the learning. Hence, there may be a causal link between teachers' self-efficacy beliefs and pupils' mathematics achievement and their self-efficacy (Ashton, Webb and Doda, 1983, Anderson, 2006, Nurlu, 2017).

Teachers with high levels of self-efficacy directly impact on the classroom by providing supportive, pupil-centred learning environments which are well managed, creative and mathematically challenging (Woolfolk, Rosoff and Hoy, 1990; Gordon, 2001; Witcher et al., 2002). In their study of the relationships between teaching experience, teacher efficacy, and attitudes towards instructional innovation Ghaith and Yaghi (1997) found that teachers with high self-efficacy were much more likely to use new teaching methods such as "the implementation of cooperative learning as a form of instructional innovation" (p. 453). In a related study, Tournaki and Podell (2005) demonstrated the impact of positive teacher self-efficacy and high teacher expectations on pupil achievement. However, contrastingly a teacher with low levels of self-efficacy was found to often exhibit negative expectations relying on unnecessarily severe punishments (Gordon, 2001) resulting in low pupil achievement and negativity about the subject. In a 40 item questionnaire survey of 70 trainee teachers' educational beliefs Witcher et al. (2002) found evidence that teacher beliefs drive educational pedagogy, and teachers with low self-efficacy often preferred to use a lecture-driven, teacher-dominated method of teaching.

2.3.2 Teacher Beliefs

Therefore, thinking about teacher self-efficacy and Witcher et al. (2002) findings led me to explore the literature relating to the beliefs a teacher holds about teaching and how these influence their classroom practice. I started to consider if self-efficacy and beliefs were one and the same, especially as Bandura (1982) had linked self-efficacy as one's belief in one's ability to succeed in specific situations or accomplish a task. A teacher's sense of self-efficacy might influence their beliefs and play a major role in how they approach the task of teaching, and the challenge of presenting the subject content. Research tends to show that teacher beliefs influence their designs of teaching activities (Nespor, 1987; Thompson, 1992; Cross, 2009; Liljedahl, 2010) and these beliefs stem from past experiences (Borko et al., 2000; Cooney, 2002; Wilson and Cooney, 2002, Borvik and Gardiner, 2007) and from their own experiences as pupils. Additionally their beliefs relating to mathematics were shown to have a direct effect on their pedagogical decision making (Skott, 2009; Polly et al., 2013). Cross (2009) also found that teachers who hold a constructivist oriented mathematical belief are more likely to adopt learner centred teaching activities than teachers who hold a more traditionally oriented mathematical set of beliefs.

However, a study by Liljedahl, Rösken, and Rolka (2006) found a misalignment amongst pre-service teachers' mathematical beliefs and their enacted beliefs with the inconsistencies being attributed to issues involving school culture and busy schedules. They also reported that the pre-service teachers held

> a belief that teaching mathematics is 'all about telling how to do it' and may come from a belief that learning mathematics is 'all about being told how to do it', which in turn may have come from personal experiences as a learner of mathematics (p. 279)

The study did establish that beliefs about mathematics subject knowledge, teaching and teaching mathematics are complex structures, are often difficult to change once formed and that any changes are tenuous and fragile. What the study was unable to demonstrate was why changes in beliefs were occurring and establish the mechanisms behind these changes. Research (Lortie, 1975; Feiman-Nemser, 1983; Ball et al., 1990; Calderhead and Robson, 1991; Craig, Kraft and du Plessis, 1998; Evans, 1999; Baliram, and Ellis, 2019) that teaching practices are informed by ideas, experiences and beliefs "that teachers begin to develop long before embracing teaching as a career and that traditional teacher preparation does not successfully challenge these beliefs" (Dembélé and Miaro-II, 2003, p. 33). Speer (2005) argues that changes between professed and attributed beliefs might be attributable to "the methods used to collect and analyse relevant data and the particular conceptualizations of beliefs implicit in the research designs (p. 361). This would appear to suggest that the problem may be that the research design is flawed, because if we do not look for problems then there is a tendency to think that they do not exist. What Speer (2005) concluded was

> whether researchers use video clip interviews or other methods for obtaining data on teachers' beliefs, it seems important for researchers to place an emphasis on developing and using methods that enable the most accurate attribution of beliefs possible instead of focusing extensively on distinctions between professed and attributed beliefs (p. 388).

What I did find interesting was that the literature tends to organise teachers' beliefs about mathematics, teaching and teaching mathematics into two broad categories. One category of beliefs tends to reflect the behaviourist, transmissionist theory of learning where learners passively receive the subject content and that the role of the teacher is to simply transmit the knowledge. The other category of beliefs tends to subscribe to the social theory of learning (eg social constructivists) and promotes conceptual understanding, problem solving and reasoning (Ross et al., 2003; Lampert et al., 2010; Voss et al., 2013). So it might be that teachers who had experienced a very didactic transmissionist mathematical education as pupils would hold the belief that this is the "correct" approach for their own practice and that it is the most appropriate way of delivering mathematical knowledge. Clark et al. (2014) in a survey of beliefs about mathematics teaching and learning, involving 59 novice upper-elementary and 184 novice middle-grades teachers, found that "knowledge and beliefs do not operate independently or in isolation; rather, teacher beliefs can act as a mediator between teacher knowledge and teacher practice" (p. 254) and that these beliefs "are strongly influenced by both socialization in the profession and experiences as students" (p. 249).

2.3.3 Teacher Subject Knowledge

So if one of the factors in a teacher's self-belief is confidence with subject knowledge I started to investigate the literature and found that quantitative studies have attempted to establish links between pupils' attainment and teacher characteristics, such as teaching-specific content knowledge, teacher professional qualification status, degree qualification, and the number of years of teaching experience (Rockoff et al., 2011). However, such studies seem unable to establish strong relationships between teachers' qualifications and pupil attainment (Wayne and Youngs, 2003; Rivkin, Hanushek and Kain, 2005; Lubienski, Lubienski and Crane, 2008).

Positive relationships have been recorded between teachers' subject knowledge of mathematics content, pedagogical approaches and pupil attainment (Hill, Rowan and Ball, 2005; Baumert et al., 2010; Kunter, 2013) as well as between teachers' beliefs about mathematics teaching and learning (Love and Kruger, 2005). Whilst, this view is not universal as there is some mixed evidence, if not contradictory, (Guyton and Farokhi, 1987; Darling-Hammond, 2000; Goldhaber and Brewer, 2000; Darling-Hammond and Bransford, 2005) all of which found both positive and negative effects of teacher qualifications impacting on pupil learning (Monk and King, 1994). Research by Goldhaber and Brewer (2000) indicates that links between formal qualifications and pupil learning might be subject specific as they found a positive relationship in mathematics, but none in science. Rowan, Chiang, and Miller (1997) report a positive relationship between pupil learning and a teacher having a mathematics degree.

Nevertheless there is some significant research that points to the way teachers teach in the classroom and the number of years of teaching experience as being the decisive factor on pupil learning (Aslam and Kingdon, 2013). They demonstrate that when a teacher's qualification and teaching strategies were investigated it was the ways in which a teacher teaches that were "more significant than fixed characteristics such as qualification" (ibid, p. 172). Similarly, the global monitoring report UNESCO (2005, p. 152) states that: 'What goes on in the classroom, and the impact of the teacher and teaching, has been identified in numerous studies as the crucial variable for improving learning outcomes" but the report does not mention academic qualifications as having an impact on pupil learning.

There also appears to be in the literature a common sense universal expectation (Shulman, 1987, 1999; Goulding, Rowland, and Barber, 2002; Hattie, 2003; Silverman and Thompson, 2008; Masters, 2009; Dinham, 2013; Zhang and

Stephens, 2013; Greaves, 2014) that all teachers should have a sound knowledge of the mathematics subject content at all levels and certainly to the level at, or above, which they are being expected to teach. In America USDE (2008, p. xxi) states that

> Research on the relationship between teachers' mathematical knowledge and students' achievement confirms the importance of teachers' content knowledge. It is self-evident that teachers cannot teach what they do not know.

and in the United Kingdom Burghes (2011, p. 17) recommends that

> A prerequisite to be an effective teacher of mathematics is that you are confident and competent in mathematics at a level significantly above that which you are teaching.

A mixed methods study examining the relationship between teacher subject knowledge and pupil attainment involving first and third grade teachers by Hill, Rowan and Ball (2005) found that subject knowledge was a significant factor in pupil achievement. In America Jacob, Hill and Corey (2017) in a three year study on the professional development of teachers' mathematical knowledge with 105 teachers in 19 low – income schools found that teachers' mathematical knowledge for teaching did "enable them to elicit more student thinking and reasoning during mathematics lessons" combined with some "limited evidence of positive impacts on teachers' mathematical knowledge for teaching, but no effects on instructional practice or student outcomes" (p. 397).

The expectation of the level of subject content knowledge by some countries is in sharp contrast to China and the Far East where mathematics teachers are required to be content specialists and are "required to study mathematics systematically and in depth" (Li et al., 2008, p. 422). Ginsburg et al. (2005) points to the superior content knowledge of Singapore mathematics teachers when compared with their American counterparts as a direct result of societal expectations rather than any formal requirements. However, a thorough knowledge of the content is anticipated by formal teaching standards documents (NCTM, 2000; QCT, 2012; AITSL, 2014) as for example in the United Kingdom teachers' standard 3 (DfE, 2012, p. 5) which states that a teacher should "demonstrate good subject and curriculum knowledge" and qualifies this with "demonstrate a critical understanding of developments in the subject".

With the National Audit Office reporting the shortages of mathematics teachers (NAO, 2016; Dickens, 2016) and the reducing percentage of those mathematics teachers with relevant degrees the Education and Training Foundation (ETF, 2014) for the United Kingdom, has been tasked with introducing the policy of Subject Knowledge Enhancement (SKE) courses for those without relevant degrees as a precursor to teacher training and as a means of increasing the pool of potential mathematics teachers. This made me think about this specific group of potential mathematics teachers and their self-efficacy and how it might be impacting on classroom teaching. A small sample survey (ETF, 2014) explored the link between teacher confidence teaching at higher GCSE level and teachers' subject qualification. This report revealed that "teachers with relatively lower subject qualifications are on average less confident in providing effective learning" (p. 4). Clarke (2011) concludes, from a small pilot study of student teachers on a 24 week mathematics subject knowledge enhancement course (SKE) using a mixed methods approach, that "teachers' beliefs of what mathematics is and, in particular, how it should be taught are tacitly formed by the way they are taught in their precollege education" (p. 7). But he also goes on to conclude that SKE courses may well have an influence over the ways in which mathematics is taught and "potentially, this is where the 'quality' of mathematics teaching could start to change" (p. 7). The paper concludes that SKE courses are likely to enhance participants subject knowledge, their confidence and hence their self-efficacy. It would be tempting to say that SKE courses are the solution to low subject knowledge but as a participant in the study noted

> It [SKE] has rekindled a passion for maths in me and I haven't had as much fun with learning for 20 years. But there is much more here than just learning maths .... There are ethical issues concerning how you teach (p. 6).

It would seem that SKE courses have much more to offer potential teachers than merely subject content knowledge.

A teacher's depth of understanding of mathematical content knowledge was shown to influence their teacher behaviours, practices and consequentially the learning of students (Koehler and Grouws, 1992; Norton, 2016). The situation is not unique to mathematics as Ormrod and Cole (1996) in a study of subject knowledge for the teaching of Geography argued that a greater depth and breadth of knowledge of subject content might result in changes of classroom practice and pedagogical knowledge. The inference here being that secure, deep subject content knowledge, influences pedagogical knowledge such as teaching procedures, processes and effective lesson planning. Crucially deep subject knowledge positively affects the selection of alternate ways of presenting the subject resulting in improved pupil attainment (Shulman, 1986; Rowan, Correnti and Miller, 2002).

The relationships between a teacher’s subject content knowledge, pedagogical knowledge and beliefs are therefore intimately related to student achievement (Muijs and Reynolds, 2015) and these can be viewed as being context specific (Fennema and Franke, 1992). Thompson (1992) theorised that because teachers treat their beliefs as knowledge, it is difficult to distinguish between knowledge and beliefs, with Manouchehri (1997) noting that teachers translate their knowledge of mathematics and pedagogy into practice through the filter of their beliefs about the subject and the classroom pedagogical procedures.

A key moment in my understanding of how mathematical content knowledge and pedagogical knowledge intertwine came when I read Ernest’s (1989a) article on the knowledge, beliefs and attitudes of a mathematics teacher. In particular the following

> Teacher knowledge, especially that of mathematics and its teaching, can be expected to influence the teacher's mathematical attitudes. It seems likely that confidence, both with regard to mathematics and its teaching, will relate to the teacher's knowledge of these areas, via the perceived adequacy of the teacher's knowledge (p. 26).

Knowledge of mathematics (content knowledge) is therefore, according to Ernest (1989a), transformed by means of practical knowledge of mathematics teaching (both pedagogical and curricular) into representations for the classroom, but the impact of teacher beliefs and attitudes are fundamental in promoting positive images of the subject. I therefore concluded that the ways in which a teacher views their own performance, their beliefs about mathematics, their perception about their own level of teaching expertise and the level of understanding of the subject knowledge all impact on the act of teaching and pupil learning. One of the dominant factors on classroom approaches to teaching seems to be not subject or teaching experience of expertise but pedagogical knowledge. This pointed me towards thinking about pedagogical knowledge in terms of the approaches that teachers adopt when planning and designing teaching sequences.

2.3.4 Approaches to Teaching Mathematics

Kuhs and Ball (1986, p. 2) identify four distinct mathematics teaching approaches

> **Learner – focused** where teaching focuses on the learner's personal construction of mathematics knowledge.
>
> **Conceptual Content – focused** with emphasis is on conceptual understanding
>
> **Performance Content- focused** with an emphasis on mastery of mathematical rules and procedures
>
> **Classroom focused** based on research about what makes effective classrooms.

I found this categorisation of the approaches to the teaching of mathematics to be both interesting and puzzling. Let me explain, interestingly all four approaches can be seen in classrooms with the second and third, in my experience, being the predominant approaches. Also the second approach is pointing towards deep teacher subject and pedagogical knowledge to promote learning. However, the first and fourth teaching approaches align with the view of developing pupil self-efficacy and consequently with the ways in which the learner views the subject and their own ability. The third approach is currently being advocated by the political view of the needs of society from mathematics teaching. Puzzlingly if research is telling us that there is a huge perception problem about the relevance and fondness for the subject in the general population why are we not advocating Kuhs and Ball's (1986) two approaches that align with pupil-self efficacy?

However, the first three approaches align with what Ernest (1989a) describes as problem solving (social constructivist) where the subject is a result of human creativity and is continually expanding. The second approach indicates a Platonist view of mathematics where the subject is a static, cohesive body of knowledge. The third relates to instrumentalist approach where we only acquire facts, rules and skills to be used for the benefit of society, what Skemp (1976) calls the instrumentalist view. The final approach assumes that mathematical content is outside the control of the teacher and their only task is to deliver the subject content in ways found to be effective as indicated by research (Thompson, 1992; Andrews and Hatch, 2000).

Research has shown that pupil self-efficacy and motivation are linked to the increased likelihood of learners engaging in advanced mathematics courses (Ma, 2006). Whilst self-efficacy and motivation are important factors when learning mathematics having the resilience to continue with difficult and complex problems has attracted considerable research. For example, research by Kooken et al. (2013) identified three factors (value: see the need to become mathematically proficient; struggle: the belief that mathematics is difficult; growth: the belief that mathematics knowledge is flexible and can grow) for "encouraging greater student participation and persistence in mathematics" (ibid, p. 2). With Lee and Johnston-Wilder (2015, p. 343) concluding that "mathematically resilient students succeed despite barriers: they are adaptive; able to cope with ambiguity; expect problems and challenges and expect to meet them successfully".

The approaches advocated in the national curriculum for mathematics in the UK (DfE, 2014) appear to be aligned to Kuhs and Ball (1986) teaching approaches. The subject is defined in terms of content but supports the view that mathematics is a creative subject (approach 1 above), is essential to everyday life, critical to science, technology and engineering, and necessary for financial literacy and most forms of employment (approach 4 above) and aims to develop conceptual understanding and the ability to recall and apply knowledge rapidly and accurately (approach 2 above). The content view of the teaching of mathematics follows from Ernest's (1989) and Skemp's (1976) outline of the instrumentalist views, with the subject content organized in line with a hierarchy of skills and concepts to be presented as sequence of steps or algorithms for pupils to master. From this perspective, the role of teacher is to "demonstrate, explain, and define the material, presenting it in an expository style" (Thompson, 1992, p. 136). Consequently, the role of students is to "listen, participate in didactic interactions and do exercise or problems using procedures that have been modelled by the teacher or text" (Kuhs and Ball, 1986, p. 23).

The statutory orders from the Department for Education and Science were that mathematics teaching approaches should be seen in terms of actions for the teacher, such as, directing and telling, demonstrating and modelling or modelling and questioning (DfEE, 2001, p. 27). Whereas research has shown (Boaler, 2002; Kleitman and Stankov, 2007; Stankov and Lee, 2008; Stankov et al., 2012; Gardiner, 2016; Perry et al., 2016; Sharma, Saxena and Singh, 2017) that defining teaching approaches in terms of problem solving applied to real-life problems can be an effective way of teaching mathematics. The encouragement of thinking rather than recall, the mastery of basics, group work, the development of reasoning and communication skills all point to improved pupil self-confidence with mathematics (Morgan, 2017).

With the introduction of the national curriculum in DES (1988) teachers and learners began to experience a major shift in the approaches being used in the teaching of mathematics. Barnes et al. (2003, p. 36) reported that mathematics "was [being] driven by a much more procedural view of mathematical learning in which understanding was being relegated in the quest for efficiency". The move to teaching approaches based on the mastery of skills and algorithms at the expense of those which allowed pupils to explore and build their own conceptual understanding was seen by Barnes et al. (2003) to be worrying. They reported that "the majority of teachers stated that the use of more extended investigational tasks had virtually disappeared from their schemes due to lack of time" (Ibid, p. 36).

Mathematics teaching approaches adopted in England over the last ten years have been heavily influenced by the introduction in 1999 of the National Strategy Initiative. Teaching has a distinctive shape and structure with learning aims and objectives shared as a matter of course in a three part lesson structure advocated as good practice by the initiative (DfEE, 2001, p. 28). The move to teaching based on episodes was an attempt to address the widely held view that the three part lesson reflected "the difficulty in practice of dividing every lesson this way and the dangers of it becoming mechanistic" (Stobart and Stoll, 2005, p. 233). Adding to this lesson structure difficulty in most schools is a timetable that is designed around short time periods which creates "difficulties for teachers in allowing pupils to finish off tasks and develop ideas" (ibid, p. 233).

Over my career I have continually considered how to perfect the art of designing an ideal lesson. Often the structure of the lesson is determined not by educational ideologies or theories relating to learning but by pure mundane considerations such as the length of the lesson the environment and the time of the day. Lesson design in terms of the language mathematics teachers used to frame aspects of lessons however is independent of these considerations and it is this characteristic of the teaching approaches used to design lessons that I now want to consider. If teacher beliefs, self-efficacy and subject knowledge all impact on the class practices I started to consider if pedagogical knowledge in the form of the pedagogical language used by teachers to define the enactment of their practice was as important.

2.3.5 The language of Pedagogical Content Knowledge

Shulman's (1986) initial description of teacher knowledge, which he calls pedagogical content knowledge, includes categories such as curriculum knowledge and knowledge of educational contexts. Matters are complicated by Shulman who openly acknowledges that over many research articles and papers he has proposed multiple lists of features whilst aiming to define teacher knowledge and that the lists of features lack any ''great cross-article consistency'' (p. 8). Shulman further acknowledges that pedagogical content knowledge is of interest because it identifies the distinctive bodies of knowledge for teaching. Pedagogical knowledge is seen as a blend of subject content knowledge and the pedagogy knowledge (Loewenberg-Ball, Thames and Phelps, 2008) so as to understand of how particular topics can be presented to learners.

> [PCK] identifies the distinctive bodies of knowledge for teaching. It represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction (Shulman, 1987, p. 8).

Shulman's view started me thinking about how mathematics lessons are often described or verbalised to pupils and other professionals in terms of aims or goals, objectives, concepts, the materials needed, prior knowledge, assessment opportunities as well as personal preferences and beliefs (Brown, 2009; Wiggins and McTighe, 2011; Amador, 2016). Thinking about this resulted in me wondering if pedagogical content knowledge was really a separate domain of knowledge or simply the refinement of the interactions between subject content knowledge and teacher experience or expertise. Certainly the level of teacher experience is a much more significant factor when designing and talking about lessons (Lederman, and Gess - Newsome, 1992; Brown, 2009).

This description of pedagogical content knowledge in terms of teacher personal preferences made me think about how teachers might converse with each other about learning sequences if they hold their own views and definitions. I was therefore interested in how teachers use language to talk amongst themselves about teaching mathematics (for example how they might describe teaching the concept of division of fractions or what language they would use to describe how they would go about teaching a skill e.g. how to divide two fractions). I could find little research into how teachers use or mediate the clarity of pedagogical language to define the learning sequences they design for pupils. It seemed to me to be a fundamental aspect of a teacher’s pedagogy that they should have a well defined and common understood professional language. Having such a language for sharing teaching ideas is needed and this may positively impact on the development of a teachers’ self-efficacy with the obvious implication for pupil understanding. So the search for common meanings for some of the frequently used terms in mathematics lessons (such activities, skills, exercises and tasks) was one of the drivers for this research and as such is the contribution to knowledge that this study offers and is explored in more detail after the next section which looks at fractions, the mathematical topic through which this study researches these pedagogical terms.

2.4 Fractions

The mathematical definition of a fraction as the quotient of two integers (where the dividend cannot be zero) compared with common place everyday definitions such as ‘a fragment’ or ‘a small bit’ helps to cause cognitive conflicts in the minds of learners. This is certainly not unique to the concept of a fraction: within mathematics other such multiple meanings exist (Pimm, 1987). The preciseness of mathematical meanings of words as compared with their less precise use in everyday language is a tension that teachers of mathematics have to recognise and plan to address in their practice. Kotsopoulos (2007) offers an explanation of this phenomenon as “students experience interference when language is borrowed from their everyday lives and used in their mathematics world" (pp. 304-305).

Add to this linguistic confusion the multiple conceptions of meanings of a single word or term and the notion that a mathematical concept could possibly have multiple views then there is greater opportunity for even further confusion in the mind of the learner. Kieren's work laid the foundations for the view that a rational number is not a single construct (conception) but rather it is better characterised as a set of related but distinct sub-constructs (Kieren, 1976, 1980, 1988, 1993). Much of the research from the prominent Rational Number Project at Minnesota University is based on the four sub-constructs suggested by Kieren (1988): a) quotient, b) measure, c) ratio number, d) multiplicative operator, with Behr et al. (1985) adding a fifth sub-construct e) part-whole relationships.

It is not surprising that given these five meanings for the term fraction as compared with the more commonly everyday usage of the word that the mathematical meanings are often misunderstood.

The quotient sub construct	This involves understanding fractions as a result of division. The fraction $\frac{1}{3}$ can be interpreted as 1 divided by 3 or the result of sharing an object among three people.
The measure sub construct	"An effective way to develop students' understanding of fractions as numbers with magnitudes is to use number lines. Number lines can clearly illustrate the magnitude of fractions; the relation between whole numbers and fractions; and the relations among fractions, decimals, and percents" (Siegler et al., 2010, p. 20).
The ratio sub construct	This involves making "a comparison between two quantities; therefore it is considered a comparative index, rather than a number" (Charalambous and Pitta-Pantazi, 2007).
The multiplicative operator sub construct	This is where fractions are used to transform numbers, it mainly involves the multiplicative aspect of fractions. For example the fraction $\frac{2}{3}$ may be perceived as finding two thirds of a given quantity.
The part-whole sub construct	This is a way of representing part of a whole set of objects or complete objects. It involves the partitioning of a shape / number of discrete objects into equal parts, (unitising) or determining how many objects would be in a whole set based on a part of the set (re-unitising).

We therefore have five clearly distinct representations of a single mathematical concept. Kieren (1976) had originally regarded part-whole relations as a separate sub-construct but later (Kieren, 1993) subsumed this relation into his measure and quotient sub-construct. These five sub-constructs have been adopted, in some form or another, by most mathematics education researchers as guide for their research. A number of researchers argue that learning and developing an understanding of fractions has a dependency based on these five sub-constructs and their interrelationships (Kieren, 1976; Behr et al., 1983; Vergnaud, 1983).

Kieren (1981) acknowledges that these five sub-constructs (part-whole, ratio, operator, quotient, and measure) are interrelated views of fractions and Behr et al. (1983) built on this work by adding a sixth sub-construct that of a decimal. This multifaceted notion of a rational number is one of the main reasons why children possibly find difficulty when learning and working with fractions (Behr et at., 1986). For clarity the five facets or alternate conceptions of a fraction is be explored in a little more detail but the visualisation of the inter-connections between the sub-constructs was explained by Behr et al. (1980) in diagram 2.4

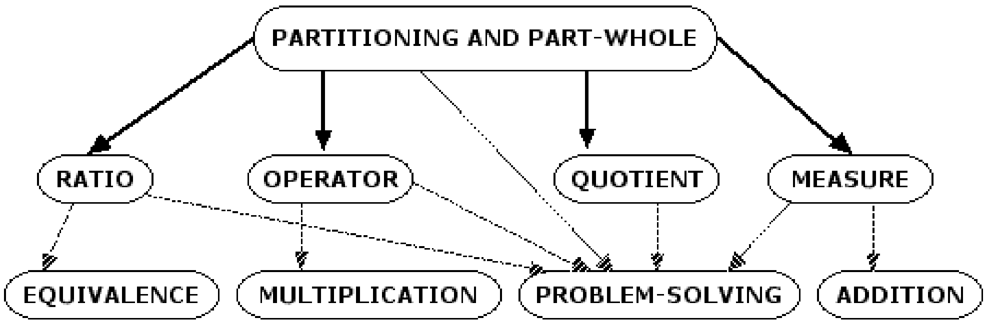


Diagram 2.4 multiple conceptions of a fractions. Behr et al. (1980)

The relationship between division and ratio and the dependence on equivalence is clearly seen from Behr et al. (1980) model of rational number partitioning. It would therefore seem logical that an approach to the teaching of division of fractions should be based on equivalency. Small (2009) describes a fractions as a relationship between a part (numerator) and a whole (denominator). Small (2009) views fractions through a number of key ideas:-

i. a fraction is not meaningful without knowing what the whole is

ii. there are always two fractions involved in any single fraction situation: the part you are considering and the rest of the whole (e.g., whenever there is $\frac{3}{4}$, there has to be a $\frac{1}{4}$);

iii. fractions can be used to describe any partitions of a whole;

iv. fractions can represent parts of regions, parts of sets, parts of measures, or ratio. These meanings are equivalent (e.g., $\frac{1}{4}$ of a region is 1 whole divided into 4 equal parts).

The first three key ideas do not sit easily with a mathematician's definition of a fraction as the division of two integers in that $\frac{9}{2}$ is clearly a fraction under this definition but has no meaning for Small's(2009) key ideas 1 or 2. A tension therefore exists between the precise, unambiguous mathematical definition of a fraction and the representations for pedagogical simplification. Fosnot and Dolk (2002) use an example such as the one below to resolve the contradiction in meanings:

> of the following fractions: $\frac{3}{4}, \frac{5}{12}, \frac{2}{3}, \frac{3}{2}, \frac{2}{5}, \frac{5}{8}$, which two fractions (and only two) will give a sum that is less than 1?

and suggest that pupils struggle with the idea that fractions can be used as comparators for part to whole relationships. When pupils are using part-whole representations for fractions Thompson and Saldanha (2003) report that students have difficulty making sense of improper fractions or mixed numbers (e.g. fractions greater than 1).

2.4.1 Fraction Visualisations

In order to exemplify these five sub-constructs this section offers five visualisations (appendix 38) based on the single, formal (neutral) and multiple representations as described by Moseley (2005).The visualisation of a fraction as part of a whole object (part–to-whole sub-construct) such as a pizza is often the preferred method used by teachers. This has the obvious disadvantage of ignoring the real mathematics meaning of a fraction as a division of integers. This method of ignoring the real meaning of a fraction often leads to difficulties when trying to divide two fractions especially as a fraction is already conceptually realised, by pupils, as a division of a whole (one).

Dividing two fractions is conceptually similar to dividing whole numbers, in that students can think about how many times the divisor goes into the dividend. For example $\frac{1}{2} \div \frac{1}{4}$ can be verbalised in terms of "How many $\frac{1}{4}$s are there in a $\frac{1}{2}$?" Teachers often use visual representations such as rectangles or a number line to help students model the division process for fractions. Students using rectangles can cut two rectangles of equal size and then separate one into fourths and one into halves. To show the division problem $\frac{1}{2} \div \frac{1}{4}$ students can find out how many fourths of a rectangle fit onto one-half of a rectangle, when the whole rectangle is the same length in both cases (Diagram 2.4.1a). This notion is based on the concept of equivalent fractions which is one of the early activities that pupils explore when beginning to learn about fractions.

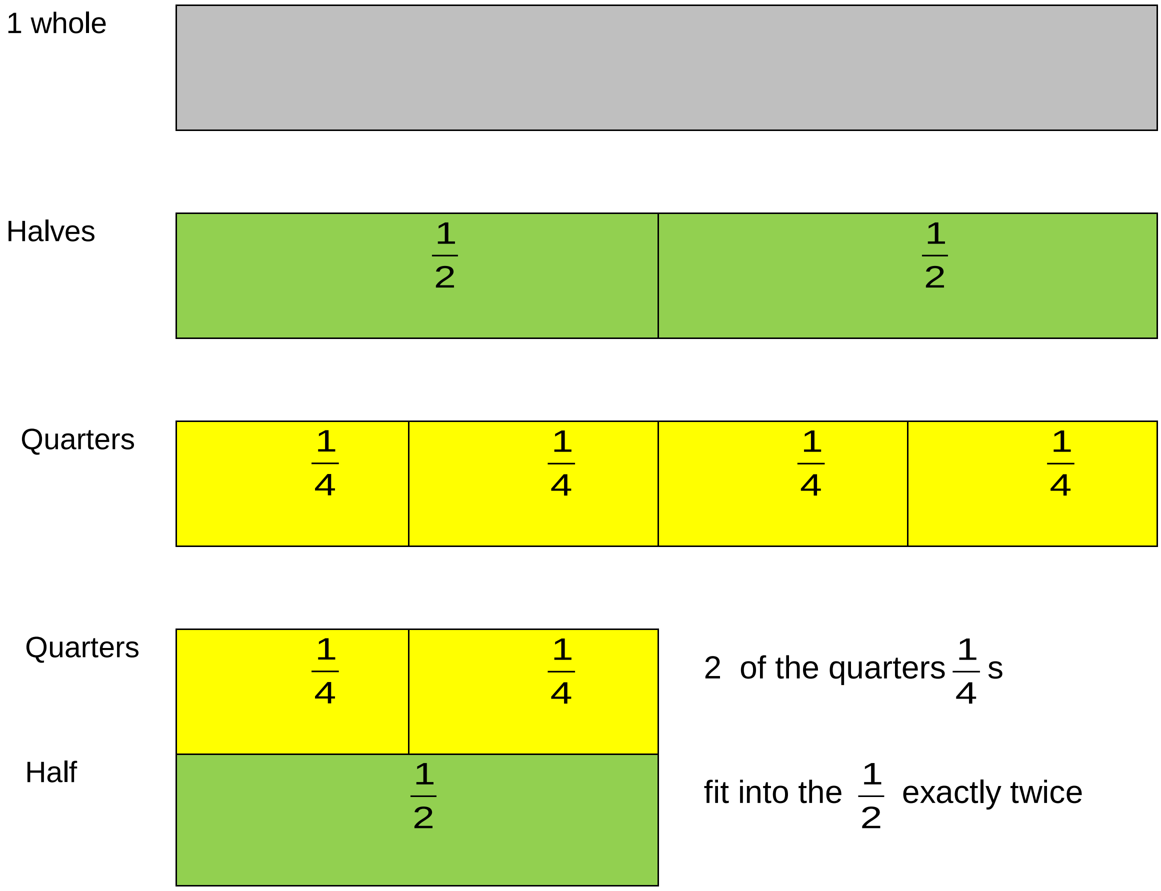


Diagram 2.4.1a Using area tiles for comparisons of fractions of a whole.

The representation of fractions in the teaching and learning sequence is often predicated on parts of pizzas or pies. Concrete examples are also often used as visual imagery and can lead to misconceptions through misuse (Resnick and Omanson, 1989; Boyd, 1992; Thompson and Thompson, 1994). The examples in appendix 49 serve to exemplify these misconceptions when dealing with fraction imagery. This all made me think why fractions are a difficult topic to learn and teach and if the approaches taken by teachers result in confusion and difficulties for pupils.

2.4.2 Learning Fractions

It is well documented in the research literature that fractions are among the most complex mathematical abstract concepts that children encounter (Davis et al., 1993; Boulet, 1998). Additionally it is widely acknowledged that fractions are difficult to teach (Hunting, 1984; Morris, 2001; Clarke, 2006). It has also been asserted that learning fractions is probably one of the most serious obstacles to the mathematical maturation of children (Behr et al., 1993). Many of the ‘trouble spots’ in elementary school mathematics are related to rational-number ideas Behr et al. (1983).

Ellerbruch and Payne (1987) investigated the role of language in facilitating mathematics learning. They report that children who, first say aloud the fraction then transcribe the oral sound before going to the symbolic form seldom make the common reversal error of writing; for example ($\frac{3}{5}$ is seldom written as $\frac{5}{3}$). There has been an emphasis on the part-to-whole sub construct in school mathematics recently and the verbalisation of fractions has largely been ignored. Children learning mathematics create frames and sub-frames as described by (Davis and Simmt, 2003) to build complex learning schemas and links between mathematical topics. Early exposure to arithmetical operations through shared dialogue creates powerful frameworks for the four arithmetic operators which can then usually be recalled and applied to other areas of mathematics. Taking for example the operation of addition once the basic framework has been discussed, shared and internalise it can be applied to the addition of decimals, algebraic expression and functions with little or no re-organisation of the framework. The application of the framework to other areas of mathematics enhances and further legitimises the framework for the learner. The difficulty arises when the learner tries to apply the framework to a topic with a different set of rules for the operation. So applying the addition framework to fractions gives erroneous results. This leads to confusion as the framework has to be altered, amended or even totally redesigned to take care of a ‘special case’, hence resulting in a higher level framework with a number of sub-frameworks to take care of the special cases'.

The clear boundaries that exist between these higher level frameworks although they can become blurred when dealing with fractions. For example current teaching approaches advocate that the division of fractions by using the algorithm of inverting and then multiplying. Whilst I recognise the interconnections that exist between the mathematical operations on the two number sets, young learners fail to grasp the interconnections. The muddling of these higher level frameworks in the eyes of the young learner creates a real tension that can result in cognitive conflict.

It is common practice when learning fractions to consider the equivalence of fractions fairly early in the learning sequence once the basic notion of fraction has been internalised by the learner. The concept of the equivalence of simple fractions is normally fairly well developed in most learners by the time they reach secondary school age. The concept is often used to deal with the addition and subtraction of fractions with differing denominators. Hence $\frac{1}{2}+\frac{1}{4}$ becomes $\frac{2}{4}+\frac{1}{4}$ and $\frac{3}{8}-\frac{1}{4}$ becomes $\frac{3}{8}-\frac{2}{8}$. Why is this approach not used to deal with the division of fractions? So $\frac{3}{8}\div\frac{1}{4}$ becomes $\frac{3}{8}\div\frac{2}{8}$ and this is $\frac{3\div 2}{8\div 8}=\frac{3\div 2}{1}=\frac{3}{2}$. The division framework learnt for integers is simply applied to fractions.

2.4.3 Teaching Fractions

Owens (1980) and Sambo (1980) examined the relationship between a pupil's concept of area and their ability to learn fraction concepts finding a positive connection. Owens found a positive relationship between success on area tasks and success in an instructional fractional unit based on geometric regions. Sambo (1980, p. 75) reports that the "deliberate teaching for transfer from area tasks aids a pupil's ability to learn fraction concepts". In contrast the findings from a study by Novillis-Larson (1980, p. 423) demonstrated that pupils working "with tasks involving the location of fractions on number lines" gained an imprecise and inflexible notion of fraction.

Since pupils actively construct knowledge, teachers must actively help them dismantle their misconceptions. Lochhead and Mestre (1988) describe an inductive technique for drawing out the contradictions in pupils' misconceptions. In the course of resolving the conflict, a process that takes time, pupils reconstruct the concept any number of times. Lochhead and Mestre (1988) suggest that there are three steps in this reconstruction process:-

1. Probe for qualitative understanding. Identify the misconception(s) and with a simple, well designed question, illicit if a pupil's difficulty comes from linguistic confusion, naive misconceptions, or both.
2. Probe for quantitative understanding. Ask for the numerical result.
3. Probe for conceptual understanding. Design the question which looks for the misconception or errors and induces conflict.

Using the example of:- Is $\frac{1}{3} > \frac{1}{2}$? These three steps might be:-

$\frac{1}{3} > \frac{1}{2}$

1. The misconception of the larger the denominator the bigger the fraction could be a result of the language used when describing fractions or more likely the positioning of integers on an increasing number line. The careful use of precise language suggested by Kajander and Lovric (2007) would turn this problem into a conversation about parts; ($\frac{1}{3}$ would be talked about as one out of three equal parts and $\frac{1}{2}$ as one out of two equal parts).
2. An approach might be to investigate the problem through equivalent fractions $\frac{1}{3} > \frac{1}{2}$ written as $\frac{2}{6} > \frac{3}{6}$ this transforms the problem into one of quantitative understanding that 2 < 3.
3. Which is larger one part out of three equal parts or one part out of two equal parts? This probes understanding in that one part out three parts of £3000 is significantly greater than one part out of two of £300.

With this inductive approach, classroom discussion can be promoted through the use of the power of dialogic teaching (Mercer, 2000). An active classroom discussion, with the teacher serving as guide, helps pupils express their misconceptions and overcome them. Misunderstandings in the use of mathematical language, both by teachers and pupils, can result in difficulties when learning mathematics. Inexact usage, by mathematics teachers, of language in defining mathematical concepts cause problems for pupils which are difficult to rectify. For example ‘When you multiply by 10 you add a zero’ or ‘All parallel lines are straight’. Once these ‘rules’ are learnt they are seldom, in my experience, ever unlearnt.

The link between exact use of language as a component of the design features of a mathematical task given to pupils and their resulting misconceptions is poorly understood, but given that task design has a relatively recent research history this is not surprising.

2.4.4 Alternate Pedagogical Approaches for Teaching Division of Fractions

By far the most widely observed and reported method of teaching the division of fractions is the ‘flip method’. This method is also the one favoured by most mathematics textbooks. It is also the algorithm that most pupils remember and in my experience the method used by over 200 students interviewed for PGCE mathematics teacher training. Just for completeness; this method is

$\frac{3}{4} \div \frac{5}{8}$ Flip the second fraction and multiply $\frac{3}{4} \times \frac{8}{5}$

$= \frac{24}{20}$

$= 1\,\frac{1}{5}$

This algorithm obviously works and gives the correct solution. What is fascinating is an often heard exchange in classrooms; such as

Teacher	Any questions before I set you some examples to try.
Pupil	Please Miss; “You can’t change a divide into a times; a divide makes things smaller and a times makes things bigger”

The algorithm described by the teacher as a means of dealing with the division of fractions (as a skill to be learnt) has caused a degree of conflict in the application of the division schema in the mind of this pupil. The algorithm does not relate to any previous knowledge or schema for how division is performed and additionally the algorithm is now using the multiplication schema. The pupil is also bringing false or partially knowledge (a divide makes things smaller and a times makes things bigger) to the argument as means of trying to resolve the conceptual conflict. The next phase of the conversation is then normally even more interesting

Teacher	“By turning the fraction upside down it allows us to change the divide into times”.
Pupil	Please Miss; “So when we turn something upside down we change it from a divide into times”.

Teacher Yes.

Again the teacher is confirming a generalisation from a specific example by agreeing with the assumption that everything turned upside down becomes and multiplication. The study of fractions in secondary school often begins with the representation of fractions as part of a whole and takes the form of either identified the fractions which are represented by shaded parts; or the shading of parts of a diagram to represent a fraction. The second step is normally the identification of equivalent fractions to be used when adding and subtracting fractions with differing denominators; after which equivalent fractions are seldom used.

I would therefore appear that the division of fractions should utilise the learning related to equivalent fractions to perform division of fractions. Taking the previous example

$\frac{3}{4} \div \frac{5}{8}$ change the three quarters to an equivalent fraction $\frac{6}{8} \div \frac{5}{8}$

Then there is no need to change the division operator and the result becomes

$= \frac{6 \div 5}{8 \div 8} = \frac{6 \div 5}{1}$

$= 1\,\frac{1}{5}$

This study adopted this approach across the attainment range. The final section of this chapter revisits learning theories and how they are interconnected when designing teaching sequences that include aspects of a lesson such as activities, skills, exercises and tasks.

2.5 The Language of Lesson Design in Mathematics

Having investigated the literature surrounding learning theories and specifically those relating to learning mathematics, teacher and pupil self efficacy and the ways in which mathematics is taught I began to wonder if language might be a factor in pupils learning mathematics. So, this section looks at the research concerning the language used to describe mathematical content and the language used to design mathematics lessons. Whilst the two languages are interconnected the research suggests that the language of mathematics is very precisely defined which is in direct contrast to the language of lesson design (Veel, 1999).

2.5.1 Lesson Design

Teachers are expected to design and teach meaningful lessons. To assist teachers in this endeavour there is an obvious need for them, over the course of their career, to be continually involved with their own learning in order to incorporate new research, ideas and theories

> Teacher learning involves developing and integrating one’s knowledge base about content, teaching and learning; becoming able to apply that knowledge in real time to make instructional decisions ; participating in the discourse of teaching; and becoming enculturated into (and engaging in) a range of teacher practices. Teacher learning is situated in teachings’ practice – including classroom instruction, planning lessons, assessment and collaboration with colleagues (Davis and Krajcik, 2005, p. 3)

A teacher’s learning is not only situated in their own practice but in discussions and collaboration with other individuals (other professional and learners) as well as artefacts and materials (Putnam and Borko, 2000). So Putman and Borko are suggesting that teachers need to participate in the discourse of teaching and the designing of lessons. Lesson designs and plans are expressed in terms of the language of the subject and the language of the professional (John, 1991). Thus, the designing of lessons requires both a knowledge of the subject content and pedagogical or professional knowledge.

The subject content of a lesson plan is often guided by either national documents or local departmental schemes. Whereas pedagogical language, or the methods by which a teacher uses to teach their lesson, are more individualistic and personal in nature and much more loosely defined. “It is therefore the professional responsibility of each teacher to apply the methods which personally suit their teaching style and take account of the ways in which their students learn” (Butt, 2008, pp. 17- 18). Even though lesson plans and designs are personal documents Butt (2008, p. 19) suggests that “the plan should be ‘teachable’ by another teacher; it should also be possible for another teacher to watch your lesson and then construct a similar plan”. An implication of this shared view of lesson plans is that the pedagogical language of the document needs to be commonly understood. The importance to the development of lesson designs and a teacher‘s practice of acquiring a pedagogical language is also highlighted by Davis and Krajcik (2005, p. 6) who state “promoting a teacher’s pedagogical design capacity can help him participate in the discourse and practice of teaching rather than merely implementing a given set of curriculum materials”.

The creative, inspirational nature of effective lessons is normally in response “to the signals and situations of the classroom in a dynamic, almost organic way” (by Davis and Krajcik, 2005, p. 35). It is therefore difficult for personally constructed lesson plans, which are then to be shared, to cater for this spontaneous element especially if there is a lack of a shared pedagogical language. However, having a shared pedagogical language about the lesson methods and procedures would allow for a more effective discourse and a mutually supportive collaborative approach to lesson planning, but it may not result in teacher’s being able to replicate each other’s enactment of the lesson.

2.5.2 The language of mathematics

I was intrigued by this apparent lack of a shared pedagogical language, especially given that mathematics teachers do use a very precise language when they are communicating mathematical ideas and concepts. Some researchers even consider mathematics as a beautiful language (Hoffert, 2009; Whiteford, 2009) but this is contested by other researchers (Hersh, 1997; Magnus , 1997; Tevebaugh , 1998). Pimm (1987), in his book on speaking and communicating mathematically, reconciles this apparent conflict when he states that "mathematics is a language in the sense that it is a metaphorical, not a literal phrase" (p. xiv). I decided that if mathematics does have a precise language then I should explore the literature around the professional language of lesson design that mathematics teachers use to see if there was a similar preciseness. I therefore wanted to explore the literature to see if my concerns or suspicion about an apparent dichotomy between the precision in the language of mathematical content and the less well defined language of lesson design did have a legitimacy in the research.

Lee (2006, p. 18) argues that mathematics is a language and that to become fluent in mathematics “pupils must be able to think in mathematical language”. Pimm (1995, p. 179) reminds us that mathematical language is used “to create, control and express their own mathematical meanings as well as to interpret the mathematical language of others”. Pupils therefore gain the mathematical vocabulary, in the main from their teachers and “pupils who are able to use mathematical language to express their ideas are able to communicate with one another and their teacher” (Lee, 2006, p. 20). Both authors are arguing that the clarity and precision of discourse is important for the development of mathematical language (vocabulary). However, the wider pedagogical language (eg. activity, skill, exercise and task) used by the teacher, to communicate mathematics to the learner may be of equal importance so that “they are able to both build and share meanings of words and expressions, and ultimately learn mathematics effectively” (Lee, 2006, p. 20).

2.5.3 Professional Language

Whilst exploring the literature about the professional language that teachers use when designing lessons, I began to appreciate that the existence of a common, well understood professional language is an on-going problem and an area for current research. The aim of this study is not to define a professional language relating to lesson design, but to try to understand how teachers speak and write about their lesson designs. I was also interested in noting differences and similarities in the language that teachers share and the impact on pupil learning (see research question 1).

Schneider and Pickett (2006) in a study involving teachers with dissimilar professional backgrounds found that differences in “professional culture and language hampered the collaboration during the enactment of the [lesson] design” (p. 12). Also Lewis, Perry and Hurd (2009) in a theoretical paper defining a model for lesson study, having worked with six mathematics teachers, suggested that “efforts to build a theoretical model of lesson study or to document the features and impact of lesson study have been modest to date” (p. 285) they concluding that

> lesson study enables teachers to strengthen professional community, and to build the norms and tools needed for instructional improvement, as situated theories of learning propose. These might include norms of inquiry and accountability and shared language and frameworks for analysis of practice (p. 286).

Cerbin and Kopp (2006, p. 254) make the very same point that “when teachers ascribe different meanings to the same basic concepts, they do not communicate effectively about the nature of teaching and how to promote better learning". In the book The Teaching Gap by Stigler and Hiebert (2009), whilst researching mathematics lesson design, suggest that the educational community in the United States "lacks a shared language for describing teaching – the way it is and the way it could be" (p. 166). Recently Zwahlen (2014), whilst studying how pre-service trainee teachers design mathematical tasks, found that the implication of a lack of a common shared language was preventing the effective design of tasks. With Pimm (1987, p. xvii) arguing that "mathematics is, among other things, a social activity, deeply concerned with communication" made me think that the professional language that a teacher uses when talking to pupils might be a key factor in the learning of mathematics.

These views relating to the difficulties of trying to find a shared professional language are not unique to mathematics teachers, especially when they are trying to design lessons, and it is certainly not a recent phenomenon. I started to consider the potential impact of this lack of a well-defined professional language on pupils who were also struggling to understand the language of mathematics content. Clements (1984), whilst investigating language factors in Australian mathematics classrooms, reported the difficulties pupils experienced with mathematical language and concluded that "children are likely to experience more language difficulties in the mathematics classroom than in any other place which they are required to attend on a regular basis" (p. 146). Here Clements (1984) is describing the difficulties children encounter in trying to understand mathematical specific terminology, if we add to that the professional language used by teachers as a means of communicating the mathematical content, then this can only compound the problem for the learner.

2.5.4 Mathematical and pedagogical language and Pupils

Gawned (1990) developed a socio-psycho-linguistic model to describe the acquisition of mathematics language. As the name of the model suggests it incorporates three components to explain how language is developed when learning mathematics. The model (figure 2.5 – section 1) proposes that children begin with the language of social interaction, as it is the foundation for all subsequent language development. Within this extensive phase of language development children develop particular components such as the language of reasoning, the language of choice, ways of describing shapes and the literacy of mathematics (e.g. dealing with numbers). Gawned (1990) also argues that children develop their language because of their interaction with their environment (the real world) and the language used by people around them. As children begin their formal education (figure 2.5 – sections 2 and 3) they are introduced to the more formal language of the mathematics classroom, both mathematical terminology and the pedagogical language of teachers. They learn the classroom discourse rules, for instance how to participate in a group or ask for help. In addition they develop the specific language for mathematical terminology. Ultimately Gawned (1990) offers the notion that the first three sections in the diagram contribute to the development and construction of mathematical meaning (section 4), at a level which the learner can independently construct mathematical meaning.

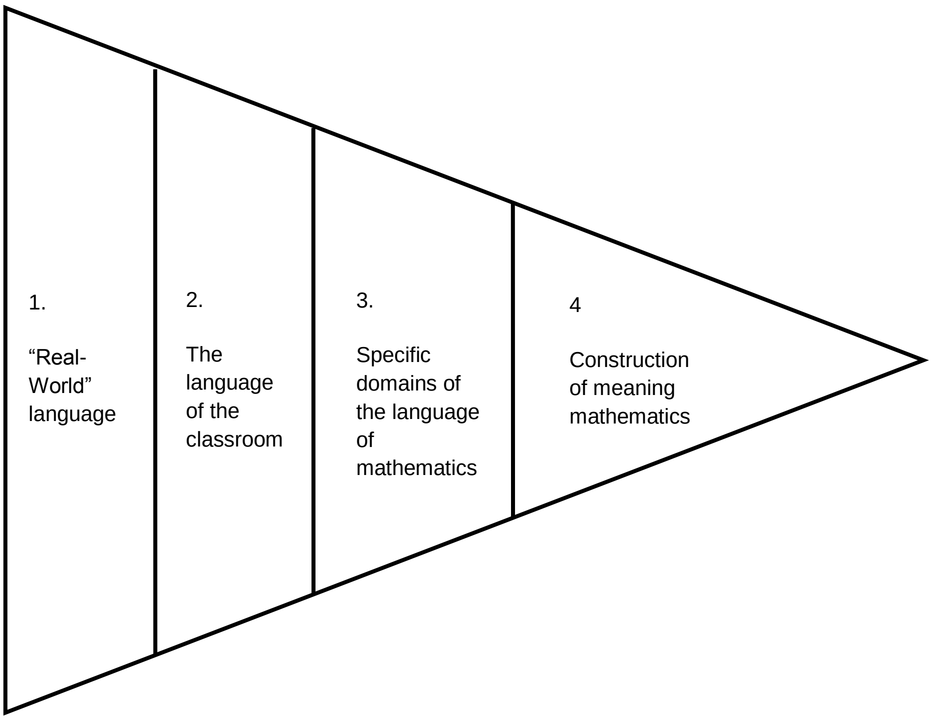


Figure 2.5. A summary of Gawned's (1990) socio-linguistic 'model' for language acquisition.

These components of language acquisition were represented by Gawned (1990) in the form of a triangle (seen in figure 2.5) to stress the development of language from the general (everyday real world) to the specific language of mathematics. The triangular shape is an indicator of the refinement of language towards the precision of mathematical meaning embodied in the content language of the subject. According to Gawned (1990) each mathematics classroom has a particular culture and language with very different patterns, rules and relationships to those experienced outside of school. An important point made by Gawned (1990) was that classroom discourse is often dominated and mediated by teachers. The socio-linguistic model does not, however, explicitly state the importance of teachers in the development of specific mathematical terminology or pedagogical language. Given the importance of a shared pedagogical language (Stigler and Hiebert, 2009; Zwahlen, 2014), it might appear that there needs to be a modification to Gawned's (1990) model to include a section relating to the importance of a common shared mathematical pedagogical language.

Particular kinds of learning episodes contained within a lesson design use specific kinds of language (Chapman, 1993). Pupils are therefore required to decipher the language of the learning episode in addition to the mathematics. Halliday (1978) uses the term "register" to refer to the category of language used in a specific learning episode and this idea of register is then described through three variables: field, tenor and mode (Halliday, 1978). Field refers to the social activity and the role language plays in the learning episode. It describes what learners and teachers are doing including a description of the topic and subject matter. Tenor refers to the roles and relationships of the learners and teachers in social learning. Mode refers to the ways and means of communication, this includes the methods and devices employed to facilitate the interactions and the ways in which learning is organized.

Of particular interest to this study is Halliday's (1978) notion of the register variable mode which Lemke (1982,) whilst investigating language in a science classroom, describes as

> The effective language of the classroom is the shared language of pupils and teachers, a constantly changing hybrid of common parlance, our ordinary ways of talking, with the registers which teachers and pupils may use in other settings (for example in textbook reading) (p. 263).

This idea of common parlance (both teacher – pupil and teacher – teacher) made me think of Zwahlen's (2014) findings that a lack of a common professional language might be a reason for the negative impact on the learning and adverse interactions with the more precise mathematical content language.

Ellerton and Clarkson (1996) undertook a substantive review of the role of language in mathematical learning reporting on factors such as the differences between 'natural' language, the formal language of mathematics, and the role of communication in mathematics textbooks. A significant finding was that the language of teaching and learning is one of the key factors that influence achievement in school mathematics. This finding had previously been reported by others such as Newman (1976) Morris (1978), Dawe (1983), Ellerton (1990) and Ellerton and Clements (1990). Garaway (1994) in a review of literature on language, culture, and attitude in mathematics and science learning reported that the mutual misunderstanding in the discourses between students, teachers, textbooks and the design of curriculum materials was a major factor in a pupils' learning of mathematics.

The role that a shared pedagogical language plays in the conversations between mathematics teachers cannot be underestimated. Using inexact professional language to express beliefs and goals about learning exemplified in lesson designs was a reason why Pajares (1992) came to the conclusion that a teacher's beliefs and lesson plans are regarded as messy items which are difficult to define. Lesson plans often refer to the detail of the actions to be taken by a teacher rather than the learning to be undertaken by pupils (Froelich, 2009), mainly because actions are easier to define and simpler to communicate. Froelich (2009, p. 1) found that the most "effective lesson designs focus primarily on designing the students' activities rather than designing what the teacher will do". These lesson designs were often less clearly expressed and contained inconsistencies in the use of language. The inconsistencies could be seen within a single lesson plan and across a group of plans, even when written by the same author.

These two pieces of research prompted me to start thinking about other aspects of shared professional language and if these aspects might be at the heart of the problems encountered by mathematics learners. For example, Foote and Kim (2015), researching mathematical academic language development in school aged children, found that teaching mathematical problem solving requires, from teachers, a precise “language to interpret the problem in context” (p. 201). They also found that problem solving “requires fluency in multiple types of language – instructional language, mathematical symbolic language and mathematical representation” (p. 201). Mercer and Sams (2006) had investigated the importance of developing the precise use of mathematical content language and the language of teaching mathematics when working with 406 children and 14 teachers in the United Kingdom. The children were solving problems which involved mathematical reasoning and found that

> the quality of dialogue between teachers and learners, and amongst learners, is of crucial importance if it is to have a significant influence on learning and educational attainment. By showing that teachers' encouragement of children's use of certain ways of using language leads to better learning and conceptual understanding in maths (Mercer and Sams, 2006, p. 26).

The results from these two investigations do demonstrate the impact of the preciseness of a mathematics teacher’s language in the development of children’s awareness and use of language as a tool for problem solving and mathematical reasoning. In addition a qualitative study by Capraro et al. (2017) compared the use of mathematical language differences among groups of 14 high school teachers whilst they were describing 3-dimensional mathematical objects. They identified three categories of language

(a) merging – language incorporating multiple types to create coherency
(b) precision – language relating to mathematical accuracy
(c) validated share meaning – language creating a shared understanding

Their findings, not surprisingly, suggested that the more successful teachers were able to produce accurate explanations of the 3-dimensional objects when they were confidently able to use all 3 categories of language. Also the study found that “teachers who used precise language were able to communicate more clearly with peers” (ibid, p. 36). The research goes on to suggest the importance of teachers using “precise language and this realization will translate into encouraging their students to work cooperatively and to complete tasks using precise language” (ibid, p. 36). Capraro et al. (2017) would appear to be connecting the precision of language used to describe mathematical content with the need for an equally precise pedagogical language.

Earlier McBer (2001), in a report for the UK government into teacher effectiveness, found that in order to solve mathematical problems a lesson needs to consist of a number of learning episodes and have a balance between teacher led instruction and pupil’s time working on problems. Lesson episodes are commonly described in terms of ‘task’, ‘exercise’, ‘skill’ and ‘activity’, all of which feature in the conversations between mathematics teachers and are used by teachers to set mathematical problems for pupils. There is an assumption by teachers that pupils might understand this language, be able to process the meaning, and act appropriately. Yet the literature suggests that such pedagogical terminology is loosely defined and not well understood by pupils (Hofmann and Mercer, 2016).

The review of the literature concerning language prompted me to think that perhaps more formal definitions of these four terms (‘activity’, ‘skill’, ‘exercise’ and ‘task’) might be useful to inform practice. These formal definitions might additionally aid the conversations and development of a precise, shared professional language. Additionally, more formal and precise understandings of the four terms that describe lesson episodes could contribute to a shared, common lesson design and hence influence pupil learning.

2.6 What is an ‘Activity?’

Whilst again there does not appear to be in the literature any clear statements as to what the features of a mathematical activity might entail, the literature does give us clues as to the various types of mathematical activities from which we may be able to deduce a generic definition for an activity. Peterson and Fennema (1985), in a number of studies researching mathematical activities, offered three specific categories of activities, competitive, co-operative and social. Their studies investigated the influences of these categories of activities on different groups of learners. They found that competitive activities are more often utilised by teachers than co-operative or social activities and that competitive activities favoured particular groups of learners. As a result of their studies an aspect of an activity might be to consider these categories as an underlying feature.

Hyde and Jaffee (1998) suggested that mathematical activities founded in social contexts might favour one group of learners over another and that mathematical social activities dependent on problem solving strategies were definitely more often suited to one gender. Sowder (1971) supported this idea and suggests that when pupils were engaged with mathematical activities based in social contexts girls may have understood problem solving strategies equally as well as the boys. However, an implication from Sowder's (1971) work was that mathematical social activities are often reliant on less mathematically abstract strategies. Many believe that student preferences are important, and that the selection of social activities might therefore inhibit the development of more abstract mathematical thinking strategies for some groups of learners.

These three categories of activities were associated with gender specific outcomes as well as relating to the different types of mathematical problems. They infer that some mathematics activities such as spatial awareness contribute to the gender differences in mathematics. Their findings also tended to suggest that whilst social activities are supportive of engagement they do not allow for the flexibility of independent high level abstract mathematical thinking and this tends to suppress attainment. Bishop (1991), therefore, argues for six types of mathematical activities

> counting : discrete aspect;
> locating : topographical features of the environment;
> measuring : continuity;
> designing : imagined form, shape, and pattern;
> playing : imagined and hypothetical behaviour;
> explaining : story telling

but again these are types of activities or even topics within mathematics and do not explicitly state the features of an activity.

Harris et al. (2010) in a comprehensive review of activities across a number of subject domains identify 31 types of mathematics specific activities which they then summarise into the following 7 categories

Activity type	Where pupils (the learner):
Consider Activities	Are asked to consider new concepts or information.
Practice Activities	Undertake computation based strategies using numeric or symbolic processing.
Interpret Activities	Consider concepts and deduce relationships between concepts.
Produce Activities	Actively produce mathematical works, rather than merely passive consumers of prepared materials.
Apply Activities	Apply mathematics to real-life situations and consider the utility of mathematics.
Evaluate Activities	Evaluate the mathematical work of others, or of their own, pupils utilize a relatively sophisticated understanding of mathematical concepts and processes
Create Activities	Develops and delivers a lesson on a particular mathematics concept, strategy, or problem.

Harris et al. (2010) have purposely placed their activities into these 7 categories as a direct result of trying to exemplify the National Council of Teachers of Mathematics' process standards (NCTM, 2000). It could be argued that practice activities are mathematical exercises and that interpreting, applying and evaluating are mathematical skills. Additionally the "evaluate" and "apply" categories of an activity are reminiscent of Ainley's (2006) features of a mathematical task.

For Dorfler (2006) mathematics is cognitive in nature where abstract objects (numbers, symbols etc) are internally manipulated. These internal representations are then externally communicated through the activities that teachers prepare for the learning of mathematics. He therefore argues that a mathematical activity must consist of two elements namely written and diagrammatic. He finally concludes:

> I can point out some features of those activities which might be of great educational / didactical impact (when organizing learning processes): Writing and reading (of inscriptions) are fundamental activities. Perceptive processes (observation, pattern recognition, inspection, comparisons) are an integral part (pp. 106 -107).

Additionally Dorfler (2006, p. 105) goes on to argue that exercises and skills are a necessary precursor for diagrammatic activities saying, "I emphasize already here an important role and function of exercise and skill in a reflected way for all kinds of activities with diagrams". Finally Dorfler (2006) concludes that activities which are based in written and diagrammatic representations might demystify and encourage learners to engage with mathematics.

As a further complication the word activities is often described by the use of a preceding adjective or phrase such as "thinking", "practical" or "mental mathematics". These types of activities are then conceived and presented to learners by teachers in the form of puzzles, or as a blank page to recall knowledge of a topic previously studied. An underlying feature of these types of activity is that they generally require little or no explanation or intervention from the teacher once they have been presented. Additionally these types of activity have their basis in social constructivist learning theory (see section 2.2) where the activity allows pupils to recall knowledge, and socially construct and reconstruct prior learning. An additional aspect of these types of activity might be the reliance on a physical object or manipulative to prompt and stimulate pupil discussions. Meira (1998) defines instructional devices, such as manipulatives, in terms of "an index of access to knowledge and activities rather than as an inherent feature of objects*. . .* a process mediated by unfolding activities and users' participation in ongoing sociocultural practices" (p. 121). Meira (1998) is therefore pointing us towards a feature of an activity being viewed in terms of physical materials, manipulatives and mathematical devices.

A mathematical activity can often be viewed as a problem that is framed in terms of its relevance to a real world context (Wood, Cobb, and Yackel, 1991). Here a real world context is perceived as the antithesis of rote learning or formulaic mathematical contexts akin to those that are often found in mathematics textbooks. Surprisingly very few mathematics education researchers or articles in the literature focus on the learning of mathematics being explored by mathematical activities that are designed to help validate mathematics as a relevant human endeavour encompassing both the classroom and real-world contexts. To summarise, an activity in the literature is

Purpose / Definition	Researcher(s)	Characteristics / Features
Competitive, Co-operative and Social	Sowder (1971) Peterson and Fennema (1985) Hyde and Jaffee (1998)	May be learner group specific
Six types of activities Counting, locating, Measuring, designing, Playing and explaining.	Bishop (1991)	Activities are described in terms of specific mathematical topics (content)
The cognitive nature of mathematics is communicated through activities	Dorfler (2006)	Exercises and skills are a necessary precursor for activities. Written and diagrams as a means of engagement and demystifying
An activity is defined in terms of the devices used (eg manipulatives) and the social practices	Meira (1998)	Basis in social constructivist learning
7 broad categories that encompass	Harris et al. (2010)	Consider, practice, interpret, produce, apply, evaluate and create.
Validate mathematics	Wood, Cobb, and Yackel (1991)	Real world contexts

Having considered the literature it has pointed me towards a definition for an activity along the lines of Harris et al. (2010) category “consider” and Dorfler’s (2006) notion of engagement and demystifying:

An Activity: is an aspect of pedagogy designed by a teacher where its main feature is that no new learning is presented, however, new learning may naturally occur. The learning aspect is designed for groups of learners to recall previously acquired knowledge, or to engage the learner, or as a hook into the next part of the learning.

2.7 What is a ‘Skill?’

The literature seems to define a multiplicity of conceptions and interpretations as to what a skill may consist of and as such these are almost always context dependant and often located in a particular situation. For example, the physical skill of passing a ball, the skill of using a calculator to perform complex calculations or the skill of playing the piano are all examples of skills and context dependant. Skills range on a continuum from those which are purely task-related such as someone’s skill at making a cake or riding a bicycle to those which are related to some natural ability such as playing a musical instrument. It can be argued that mathematical skills encompass the entire continuum.

Lowe and Muller (2010) define a skill in terms of what can be done or what can be tested: “a skill is both a modal notion (what somebody is able to do even while not doing it) and has an empirical side (skills can be tested)” (p. 265). Combining Lowe and Muller‘s definition with Dreyfus and Dreyfus (2004) five stages of skill acquisition model Novice, Advanced Beginner, Competent, Proficient and Expert (table 2.7) we are able to explain how skills might be developed.

Novice:	Application of context-free rules through information processing.
Advanced Beginner:	Application of rules also based on perceived similarity with prior examples.
Competent:	Application of a hierarchical procedure of decision making (problem solving).
Proficient:	Deep involvement, experiencing situations from a perspective “holistic similarity recognition”; Dreyfus and Dreyfus (2004, p. 28); decisions grounded analytically.
Expert:	No need for rules. They normally do not solve problems or make decisions they do what normally works.

Table 2.7 Dreyfus – Dreyfus (2004) model of the Five Stages of Skill Acquisition

Interestingly whilst the Dreyfus-Dreyfus model was empirically grounded and employed to describe professional skills such as those acquired by nurses, it is strange to note that the model was not applied to mathematical skills especially given that Herbert and Stuart Dreyfus are a mathematician and philosopher respectively. Lowe and Muller (2010) claim that mathematical skills can be viewed in terms of those skills which a professional mathematician might use. They argue for mathematical skills in terms of knowledge “that” (eg that 4+3 = 7) and knowledge “how” (eg how to convert metres into feet). They theorise that

> in the process of mathematical research, a lot of skills are involved in a successful research episode: a mathematician tackles a research question, asks the right people who give her ideas helping on her way to the correct proofs, finally finds the proof, writes it up in a way that she can communicate it to the experts, gives a number of seminar talks on the proof, receives comments from peers in these talks, fixes a number of inaccuracies and uncertainties in the proof, types a journal paper, submits the paper, goes to international conferences reporting on the result, receives a referee report with revisions, revises the paper, and finally publishes it (Lowe and Muller, 2010, p. 272).

Lowe and Muller’s 2010 view of mathematical skills is interesting and entirely recognisable by a teacher of mathematics, but their view may not entirely relevant to the general population and this can create tensions for the teacher.

However, employers often require employees to possess a different set of mathematical skills from those above such as numeracy, reasoning and problem solving (DfBIS, 2016) which enable individuals to build constructive working relationships. For employers mathematical skills are embedded in a wide range of other softer skills, which might include

1. Communication skills
2. Decision making skills
3. Self-motivation skills
4. Leadership skills
5. Team-working skills
6. Thinking skills
7. Time management skills and ability to work under pressure

(Skillsyouneed, 2017)

All of the soft skills above have at least one common feature that of being dependent upon an action. Taking thinking skills as an example and exploring more deeply this soft skill encompasses two broad categories; cognitive and metacognition including Bloom's Taxonomy, DeBono's thinking tools and Lipman's modes of thinking (Moseley et al., 2005), so soft skills have a tendency to be linked with underlying features and attributes.

So where do mathematical skills lie in the above spectrum? The argument from the national curriculum DfE(2013) and Skillsforyou (2017) is that the soft skills are developed alongside, and in conjunction with, those more recognisable mathematical skills. Returning to Lowe and Muller (2010) and their suggestion that mathematics skills be viewed “as professional skills” leads them to conclude that if professional skills can be tested in examinations then they “may well be that the aim of mathematics education is best characterised not as instilling mathematical knowledge, but as teaching mathematical skills” in order to pass examinations (p. 266). So rather than mathematical and soft skills being simultaneously developed the teaching of mathematical skills often takes precedence because of examinations.

This led me to think about examinations as assessments of skills and the often heard phrase “teaching to the test”. However, research does show that merely teaching to the test creates surface or instrumental learning rather than deep, connected relational learning (Skemp, 1976) and therefore “may not adequately capture students’ mastery of the mathematics” (Jennings and Bearak, 2014, p. 387). So a mathematical education reliant on teaching skills as defined by Lowe and Muller (2010) may not be in the best interests of learners.

The UK mathematics national curriculum identifies the following skills that should be developed in learners:-

Problem solving skills
Calculation skills
Algebraic manipulation skills
Graphing skills
(DfE, 2013)

These mathematical skills do appear to be fairly self-contained or perhaps related to specific mathematical topics. Even this list is not universally accepted and the UK national curriculum documents do not give a definitive definition of what constitutes a mathematical skill. In the 1970s there was a general acceptance that basic mathematical skills (appendices 40 and 41), fall into a number of categories which are fundamental to pupils’ mathematical development these being:-

1. Problem Solving,
2. Applying Mathematics to Everyday Situations,
3. Alertness to the Reasonableness of Results,
4. Estimation and Approximation,
5. Appropriate Computational Skills,
6. Geometry,
7. Measurement,
8. Reading,
9. Interpreting and Constructing Tables, Charts, and Graphs and Using Mathematics to Predict,
10. Computer Literacy.

(NCSM , 1977, pp. 4-6)

Some recent attempts to formulate a definition of mathematical skills in terms of features and attributes has been attempted, for example, the influential expert report “Maths Counts” by Smith (2004) has over 200 references to the phrase “mathematical skills” but no single helpful clear unambiguous definition. Other expert reports have tried to come to a conclusion about mathematical skills by trying to define skills in terms of the needs of society

> At the one extreme are those young people for whom mathematics is about the most basic of life skills, like telling the time and counting the money in their pockets. At the other end of the spectrum there are those who will go on to create and work with some of the most sophisticated ideas known to mankind (Vorderman et al., 2011, p. 17).

Earlier, Cockcroft (1982) had offered a number of definitions of mathematical skills purely in terms of the needs of employers, for example, in the manufacturing industry mathematical skills might include being able to add, subtract, multiply and sometimes divide whole numbers, perhaps with the help of a calculator (p. 35). As can be witnessed from this definition the context is all important and the implication is that the definition of a skill then becomes context dependant. But given society is developing at an ever increasing rate this type of skills definition is likely to become outdated and even obsolete. As Stigler and Hiebert (1999) report, defining skills in this manner leads to mathematics teaching consisting of large periods of time spend acquiring isolated mathematical skills through the repeated practice of similar banks of questions (appendices 40 and 41).

Defining mathematical skills in terms of learning intentions (AfL, 2017) which include what pupils should know (the mathematical knowledge) and understand (the mathematical understanding) might be a better way forward. Thinking of a mathematical skill in terms of learning intentions might also correlate much more directly with the design and planning of lessons. As a means of exemplification this might be best explained through a standard problem in the secondary mathematics curriculum, that of knowing how to construct linear graph (for example $y = 3x + 2$).

For Lehtinen et al. (2017) mathematical skills are intimately related to the development of conceptual and procedural knowledge (see section 2.2). The two types of knowledge are bound to the construction of deep learning (conceptual) and a sequence of steps or actions (procedural). Procedural knowledge is often developed and enacted by the repetition of routines and drills to be practised (Hiebert and LeFevre, 1986; Baroody, 2003). Lehtinen et al. (2017) argue that “mathematical skills in educational contexts can be characterized as drill-and-practice that helps automatize basic skills, but often leads to inert routine skills instead of adaptive and flexible” (p. 1). Many studies equate the acquisition and facility of mathematical skills with drill-and-practice type questions (Tournaki, 2003, Fuchs et al., 2010) and that these can be developed by learners through interacting with computer programs. However, according to Ericson (2016), the practising of questions has to be deliberate, planned and supported by knowledgeable people such as teachers.

A mathematical skill could be presented to the learner in the form of a worksheet or a set of practice questions designed by the teacher which might be taken from a textbook. It is a closed piece of learning which is fairly tightly defined and controlled by the teacher and requires an exposition or explanation that might be followed by a teacher intervention. A skill has its basis in behavioural learning theory and described in teaching plans in terms of what pupils might be expected to do by the end of the learning episode.

A mathematical skill viewed in terms of a learning intention as advocated by AfL (2017) might, for example, focus on how to construct a graph. The mathematical skill in terms of a learning intention would then be a combination of knowledge (graphs) and understanding (how to construct). This wider view of a mathematical skill in terms of a learning intention might enhance the transferability of mathematics skills both within, and across, different employment sectors and academic disciplines (Britton, 2002). However, it is not a view that most teachers would recognise and so it was not adopted for this study.

Summarising the literature on skills:

Purpose / Definition	Researcher(s)	Characteristics / Features
Generic definition in five stages	Dreyfus and Dreyfus (2004)	Hierarchical list of development features Novice, Advanced Beginner, Competent, Proficient and Expert
Professional skills Knowing "that" and "how"	Lowe and Muller (2010)	Tested in examinations
Soft Skills	DfBIS (2016) Skillsforyou (2017)	Eg. Communication, Decision making, Self-motivation, Leadership Team-working, Thinking Time management.
Defined in terms of mathematical content	NCSM (1977) Cockcroft (1982) Smith (2004) Vorderman et al. (2011) Mathematics National Curriculum (2013)	Eg Computation, geometric interpreting graphs and tables.
Learning intentions	AfL (2017)	Mathematics they know and understand
Conceptual and procedural knowledge	Lehtinen et al. (2017)	Drill and practice

The review of literature has therefore pointed me towards a definition for a skill as being

A Skill: is an aspect of pedagogy designed by a teacher where a new piece of mathematical knowledge is presented and the learner is required to complete a number of prescribed questions to practise and perfect the new learning, an example would be the division of two fractions.

2.8 What is an 'Exercise?'

A common understanding of the term exercise, when related to mathematics teaching, might be that of a number of textbook problems to develop a particular mathematical skill or set of skills. However the word exercise does have a multiplicity of meanings such as a physical exercise, the exercise to practice a skill or an exercise as a piece of work intended to test knowledge. Even when we consult a single source such as the 2002 Proceedings of the Annual Meeting of the Canadian Mathematics Education Study Group four contributors use the word in different ways for example:

> Once found, it is easy to prove in Mathematica, in Maple or by hand—and provides a very nice calculus **exercise** (Borwein, 2002, p. 24).
>
> Participants are first asked to make note of geometric shapes that they notice around them—a task that has consistently and reliably given rise to lengthy lists of Euclidean forms. That list is pushed aside during the fractal cards activity, after which participants are invited to repeat the **exercise** of looking for geometric shapes (Gerofsky, Sinclair, and Davis, 2002, p. 37).
>
> This led the group to consider if there is an empirical way to study this question of emotions in mathematics? For example, divided-page **exercises**. Give them a problem and on one side they do the problem and the other side they write thoughts and ideas. Those teachers were using the **exercises** to help them work on their emotions (Pallascio and Simmt, 2002, p. 49).
>
> My goal is to pursue this and investigate conceptualizations across contexts, such as geometric, graphical, analytical, and others, and also across approaches, such as responding to situations, discovery **exercises**, constructions and counterexamples to force focusing on properties and definitions, and to look for themes and patterns (Brown, 2002, p. 73).

Borwein (2002) appears to be using the term exercise to describe a learning episode related to a single problem that investigates a mathematical concept. This is in contrast to Gerofsky, Sinclair, and Davis (2002) who are using the term to describe a repetition of a piece of learning once additional mathematical information has been added. For Pallascio and Simmt (2002) the term exercise is used to describe a method of presenting a mathematical solution and the effect that the resulting presentation has on the emotions of the learner. Brown (2002) is focusing on the use of the term exercise as a means of a learning episode described through mathematical discovery.

All four researchers would therefore appear to be using the word exercise in different ways. I decided to consult more sources and found that teachers and textbooks tend to use the term exercise as a means of demonstrating mathematical procedures, algorithms or worked examples as a precursor to the setting of a repetitive sequence of questions for pupils to undertake as reported by Post et al. (1993, p. 1) view of textbooks as

> Page after page of drill and practice exercises are still the norm rather than the exception; problem solving seemingly has more to do with the

> existence of words than it has to do with the presence of a problematic situation for which the person involved has no readymade response patterns - the more or less standard definition of problem solving. The presence of real-world problem situations that will require extended and repeated periods of contemplation are virtually non-existent.

Hattie (2012) argues that one of the roles of a school and teachers is to teach pupils the value of deliberate practice (exercises that involve challenge, concentration, monitoring and instant feedback). This is in comparison to the more normal view of an exercise as the practising of a set of repetitive similar questions (Rasmussen et al., 2005). Here Hattie is not trying to define the term excises but simply to describe an exercise in terms of its features and qualities. Similarly the features of a mathematical exercise were investigated by Lithner (2003) who found that of 600 exercises the majority were possible to solve using method such as identifying similarities in between questions which he calls “Identification of Similarities” (ibid, p. 35). However, on closer inspection the research uses the word “exercises” to mean mathematics questions that are either designed by teachers or taken from textbooks. Clements (2000) presents the more traditional view of an exercise as “drill-and-practice” (p. 10) but it is duly noted that “students need more than drill and practice; they need to understand the mathematical concepts beyond the practice exercises (Davis et al., 1990). So to summarise the various uses of the term exercise:-

Purpose / Definition	Researcher(s)	Characteristics / Features
Investigating mathematics	Borwein (2002)	Engagement
Repetition of learning	Gerofsky, Sinclair, and Davis (2002)	Additional information being added to create deeper levels of understanding
Presenting a mathematical solution	Pallascio and Simmt (2002)	Emotional
Discovering mathematics	Brown (2002)	Discovery
Demonstrating mathematics	Post et al. (1993)	Procedures, algorithms or worked examples
Deliberate practice	Hattie (2012)	Challenge, concentration, monitoring and instant feedback
Drill-and-practice	Davis et al. (1990); Clements (2000)	Practising and creating a deeper mathematical understanding of concepts
Identifying similarities	Lithner (2003)	Designed by teachers

With a number of differing views in the literature as to what constitutes or defines a mathematical exercise I decided that because the study would involve practising teachers I would need to use a definition that would be both recognisable to them and be true to the aim of the study. I therefore decided that Clements' (2000) view of an exercise as drill and practice would be recognisable to teachers and that Brown's (2002) view of learners constructing learning by discovery would be the feature that this study would investigate. The idea of pupils discovering mathematics by constructing the learning through posing their own questions is seen by a number of mathematics education researchers as an important aspect of learning (Kilpatrick, 1987; Silver, 1994; Chin and Kayalvizhi , 2010; Abramovich, 2015; Wong, 2015). At the point where the learner takes control of the learning the teacher's role is to loosely guide rather than direct the learner. In this respect it is an open piece of learning which requires little exposition and explanation and usually follows pathways which the learner is interested in pursuing. An exercise therefore has its basis in self-directed learning and experiential learning theory (see section 2.2) where the exercise allows pupils to socially construct learning (see section 2.2) based on their experiences.

The literature therefore leads me to my working definition of an exercise using as the basis Clements (2000) and Silver (1994) and is defined as follows

An Exercise: is an aspect of pedagogy designed by a teacher to encourage learning and is seen as an extension to drill and practice type questions to gain an understanding of a mathematical concept. Additionally the newly acquired piece of learning is explored through a limited number of teacher prescribed questions and importantly extended by an additional set of learner generated questions.

2.9 What is a 'Task?'

The definition of a 'task', in the learning and teaching sense, is a piece of work imposed by the teacher is fairly loose and ill-defined (Littlewood, 2004). For many teachers this looseness of definition presents little or no problem when discussing their professional practice. For other researchers a task is viewed in terms of an action "that learners engage in to further the process of learning" (Williams and Burden, 1997, p. 168). Collins (1996) had sought to define tasks using four parameters (appendix 49) with a range of exemplars for each, but these parameters were generic in nature and could be equally applied to other types of learning episodes such as activities. A much more detailed definition of a task by Breen's (1987, p. 23) as "a range of activities from the simple and brief exercise type to the more complex and lengthy activities such as group problem-solving or simulations and decision making". The use of activity and exercise to define a task is confusing because Bren may be using the words activity and exercise in a physical sense or as mathematical pieces of work. In other subjects such as the teaching of language there seems to be little or no distinction between a task and an activity as Willis (1996, p. 53) explains that "by `task' I mean a goal-oriented activity in which learners use language to achieve a real outcome".

The term 'task' has been used for years amongst mathematics teachers and educators, in staffrooms and policy documents (Cockcroft, 1982, NCTM, 2000, DFFE, 2001) when describing aspects of a lesson design. None of these documents over the years has given us a concise working definition. The precise meaning of the term task in mathematics teaching would therefore appear to be problematic or left to the teacher to loosely interpret for themselves.

Exploring the literature in more depth for well-defined meanings resulted in Leont'ev's (1975) view of a task as an operation to be undertaken which is bounded by constraints and conditions and Chevallard's (1999) view as techniques and aspects of a human activity. Contrastingly Mason and Johnston-Wilder (2006) view a task as what pupils are being asked to do with Becker and Shimada (1997) defining tasks in terms of the materials or the environments which are intended to promote complex mathematical activity for example "rich tasks" (Swan, 2006). We also know from Artigue and Perrin-Glorian (1991) that tasks are at the core of mathematics education and that tasks generate activity, develop mathematical thinking and lines of enquiry. I found this last statement very interesting in that a task leads to activity and therefore the terms cannot be equivalent as others would seem to imply.

The international Commission on Mathematical Instruction (ICMI) study 22 (2013) devoted the whole conference to Task Design in Mathematics Education. The conference proposed the working definition of a task to be "anything that a teacher uses to demonstrate mathematics" (ICMI, 2013, p. 12). The notion was further developed by Coles and Brown (2016, p. 150) whilst working with a schools' mathematics department who suggested that a mathematical task be

> anything that a teacher uses to demonstrate mathematics, to pursue interactively with students, or to ask students to do something. Tasks can also be anything that students decide to do for themselves in a particular situation.

This notion of a task being teacher generated is at the heart of Simon's (2013) broader definition "as either a question that a student is asked or an objective that they are given to accomplish (e.g. Why is a triangle the only rigid polygon?)" (p. 504). From this I inferred that tasks should encourage a pupils' self-efficacy or their self-belief and include an element of exploring mathematics. To some extent the view of Watson and Ohtani (2012, p. 4) that a task is "anything that students decide to do for themselves" again appears to support the idea of pupil self-belief as they are determining the mathematics that they personally want to experience. Previously Doyle (1983) had defined tasks in terms of a product that students are to formulate, such as the answers to a set of questions, again indicating the notion that tasks are posed and generated by pupils for self-exploration. Lee (2006, p. 55) also argued that asking pupils "to write their own questions to form a similar, but possibly more interesting" problem was beneficial.

There is therefore a possible tension here between tasks being defined by a teacher and or a pupil. However all the research indicates that mathematics is to be considered a social endeavour and therefore tasks being defined by both teachers and pupils is consistent with this notion. For Barbosa and Pereira de Oliveira (2013) this apparent tension does not create a problem as they explain:

> Mutual learning does not remove horizontal hierarchies among the participants. Let us imagine a collaborative group in which a well-respected academic takes part in that. In such a case, the voice of the academic may be stronger in the group than other participants. The same may happen to an experienced teacher who has a higher expertise about anticipating students' actions (p. 545).

Yet another alternative way of defining a task is offered by Watson et al. (2008, p. 1) and is stated in terms of the types of mathematical thinking stimulated in pupils by tasks

> which enable learners to make shifts in understanding which are central to secondary mathematics. For example, we were interested in shifts from additive to multiplicative (and exponential) thinking.

Ingram and Ward-Penny (2010) warn us that a pupil might be 'engaged' with a task without actually engaging with, or even making shifts in, their thinking about the mathematics that they are exploring. Ingram and Ward-Penny (2010) also found that tasks based on practical equipment or games might guard against this lack of engagement. Tasks with these underlying features also stimulated thinking which they found to be a requirement for pupils to articulate their thinking both in writing and in speech.

In 2011 at the Monash University, Australia a research project was designed to investigate ways of classifying tasks in terms of their type. The project eventually recommended that tasks could be grouped into four distinct categories

> Type 1: A task used to model mathematics
> Type 2: A task with the mathematics set in a practical context
> Type 3: A task as open-ended investigation
> Type 4: A task as a multi-domain (interdisciplinary) investigations to explore both mathematics and the other domains

The findings of the project relating to type 3 tasks did confirm the study by Stein and Lane (1996) that student performance was enhanced and encouraged by open-ended tasks. Boaler (2002) had also previously compared the outcomes from working on open-ended tasks in two schools. In one school the teachers based their teaching on open-ended tasks and in the other on traditional textbook based approaches. Boaler (2002) found that the students in the school adopting the open-ended approach “attained significantly higher grades on a range of assessments, including the national examination” (p. 246).

From the literature there also seems to be an indication that tasks have an underlying aim of changing or enhancing conceptual understanding, fluency and mathematical accuracy. This is definitely the view that Kilpatrick, Swafford, and Findell (2001) had in mind when they described learning mathematics as multi-faceted leading to proficiency when pupils work on a wide variety of mathematical tasks. This view is supported by Boston and Smith’s (2009, p. 136) definition of a mathematical task as “a single complex problem or a set of problems that focuses students’ attention on a specific mathematical idea”. A wider definition of a mathematical task from Watson and Mason (2007) indicates that a task “in the full sense includes the activity which results from learners embarking on a task" (p. 207), but they additionally argue that this definition of a task is not taken-as-shared in mathematics education.

It might be therefore argued that this range of views and definitions of a mathematical task evident in the literature could lead teachers to using the term in very different ways and confusing it with other pedagogical terminology. This is evident for example, when Watson and Mason (2006) define an exercise to be an object or a "collection of procedural questions or ***tasks***" (p. 91, my emphasis) thereby connecting an exercise with a task. Later in the paper they connect both lessons and activities with tasks, stating "Learning cannot generally be predicted or identified in discrete chunks of time, say over one activity, or one lesson, or in a particular task sequence" (p. 92). Furthermore they define a task in terms of a "modelling activity" and then state that "Opportunities to practise skills, to select and represent variables, to express relationships and generalities, to gain mathematising tools with which to engage economically and critically with the world, are overwhelmingly present in modelling activities" (Watson and Mason, 2006, p. 108). Thus they are connecting skills with tasks, tasks with exercises and exercises with activities. This interchangeable use of the pedagogical terms task, exercise, skill and activity might create a degree of confusion in the minds of teachers.

So as a result of reading Watson and Mason's (2006) research it led me to question the possible confusions or contradictions as they described a task as a "modelling activity", a skill and an exercise. Even more confusingly they go on to associate the practising of skills and exercises, which are often synonymous in mathematics particularly at school level with questions found in textbooks or on worksheets. This is in contrast to tasks at school level which are more likely to be associated with open-ended problems and pseudo mathematical investigations. So, is a task an exercise, a skill or is it an activity or does it really matter? If one mathematics teacher is using the term 'task' to describe a set of procedural, textbook questions and another teacher to describe an open-ended investigation, then I conjecture that this could lead to confusing conversations. Additionally this confusion in the use of pedagogical language might not support shared lesson planning, or Butt's (2008) view that lesson plans should be able to be delivered by another teacher.

Leinhardt, Zaslavsky and Stein (1990) investigated a range of different types of tasks and classified them into two broad categories; as an action taken by the learner or as a construction undertaken.

> By action we refer to first whether the task is interpretation (e.g., reading, gaining meaning) or construction (e.g., plotting a graph from a data set, determining an equation from a graph, or generating an example of a function) (p. 4).

The notion of defining a task by its features was further developed by Ainley (2006), Ainley, Pratt and Hansen (2006), Margolinas, 2013 and Ainley and Margolinas (2015) who propose tasks should have two features: these being *purpose* and *utility*. Ainley (2006) defines *purpose* in terms of “the perceptions of the pupil rather than to any uses of mathematics outside the classroom context” and this “may be quite distinct from any objectives identified by the teacher” (p. 1). This leaves the *utility* of a task to be defined in terms of “how, when and why that idea is useful” (Ainley, 2006, p. 3). This distinction is useful as it allows teachers to view school mathematics through the perspective of not only solely being concentrated on practising basic skills (Burton, 1984; Lampert, 1990) but as a way of developing mathematical thinking and the usefulness of the subject, which may contribute to a pupils’ self-belief in their mathematical ability. Ainley’s (2006) definition of a task according to features *purpose* and *utility* might therefore legitimise a teachers’ view of practising procedures or solving ‘word problems’ as just two of the many different variants of a mathematical task.

Summarising the arguments and sorting them into groups the literature is pointing towards

Purpose / Definition	Researcher(s)	Characteristics / Features
Action “that learners engage in to further the process of learning” Problems that pupils can articulate mathematical thinking both in writing and in speech.	Williams and Burden(1997) Ingram and Ward-Penny (2010)	Engagement
A range of activities from the simple and brief to lengthy activities	Breen (1987) Artigue and Perrin-Glorian (1991) Chevallard (1999)	Developing problem solving, decision making, thinking and enquiry
Goal-oriented	Willis (1996)	Use of language
Purpose / definition	Researcher(s)	Characteristics / features
A type of mathematical thinking	Watson et al. (2008)	Makes shifts in pupil understanding
Multi-faceted and varied problem	Kilpatrick, Swafford, and Findell (2001)	Mathematical proficiency
An operation	Leont'ev’s (1975)	Bounded by conditions and constraints
A task is what a pupil is required to do	Mason and Johnston-Wilder (2006)	
Open-ended problem.	Stein and Lane (1996) Boaler (2002) Monash University project (2011)	Enhanced performance / assessment
Anything that a teacher uses to demonstrate mathematics, or anything a pupil decides to explore	Coles and Brown (2016)	Teacher or pupil generated
A question that a pupil is asked or an objective that they are given to accomplish	Simon’s (2013)	Teacher generated
Anything a pupil decides to do for them self. A product that a pupil generates such as the answers to a set of questions.	Doyle (1983) Watson and Ohtani (2012)	Pupil generated
A single complex problem Promotion of complex mathematical activity	Becker and Shimada (1997) Boston and Smith (2009)	Specific focused mathematical idea
Two broad categories Action to be taken or a construction	Leinhardt, Zaslavsky and Stein (1990)	Action to be taken or a construction of the mathematics
Two features Purpose and utility	Ainley (2006), Ainley, Pratt and Hansen (2006) and Ainley and Margolinas (2015)	

Therefore, the literature concerning current knowledge of the meaning of a task does begin to indicate a consensual view in terms of the features that inspire good mathematical tasks. However, I would suggest that this may be because the studies above have unanimously concentrated on task design as exemplified by research based on specific single types of mathematical topics or problems. Looking at the features of a task, as described over many mathematical topics, such as those found in a scheme of work might illuminate the wider generic features of tasks.

Ainley's (2006) original two features of a task (utility and purpose) together with three additional features of promoting engagement for the development of mathematical knowledge, pupil – teacher generated problems and the promotion of language in the form of purposeful talk (Willis, 1996; Ainley, Bills and Wilson, 2004) could give a broader definition of a mathematical task. Take for example the mathematical task in appendix 42, a typical mathematical task found in the secondary school curriculum, it may not meet the definitions of purpose and utility but would definitely meet these other three features.

Having reviewed the literature concerning mathematical tasks I now offer my definition and it is the one to be used in this study.

A Task: is an aspect of pedagogy designed by a teacher to prompt the application of newly acquired mathematical knowledge or learning to either a real-life problem (purpose) or a contrived but messy situation and has a usefulness (utility). A task is a piece of learning which requires some exposition and explanation from the teacher and often follows a prescribed pathway but has the opportunity for pupils to define their own problems and promotes mathematical discussions. The mathematical knowledge required for the task will have been acquired and developed in the exercises, activities and skills part of the lesson.

So with mathematics educational researchers suggesting we frame the learning of mathematics in terms of tasks (Swan, 2005, 2006; Ainley, 2006; Watson and Mason, 2006; Rich, 2018) the question arises "do skills, exercises and activities have any place in the learning of mathematics and the planning of lessons?"

2.10 Lesson Planning

Designing lessons has always been part of the everyday practice of a teacher. This started me thinking about whether teachers in all countries plan mathematics lessons in the same way. If they do plan in similar ways is the lesson structure determining the types of learning episodes? If so is there a particular type of lesson design structure that is the most effective.

In the UK, the Key Stage 3 National Strategies Initiative (DfEE, 2001), advocated a mathematics lesson model consisting of the following four parts and two features, respectively: Objectives, Starter, Main Activity, Plenary, Vocabulary, Resources where lessons are framed in terms of blocks of time (learning episodes) or as a sequence of items to be covered in a set period of time in order to achieve a particular learning objective(s). This type of lesson structure gives the impression for both the teacher and pupil that mathematics learning is the same each day and “the same for all students—operationally defined as exit behaviours and measured against a system of national bench-marking” (John, 2006, p. 484). This lesson planning model, framed in terms of a prescribed sequence of actions, might also be considered to be restrictive and “in turn misrepresent the richer expectations that might emerge from a constructive and creative use of curriculum documents” (John, 2006, p. 484). The restrictive, mechanistic nature of this type of lesson design might also be a reason why UK mathematics teachers have developed their practice in particular ways. Additionally the approach advocated might be preventing professional discussions about lesson design as the prescribed methodology is seen to be the de facto model, hence negating the need for discussion and the development of shared meanings (John, 2006).

I wondered if this rigid, mechanistic model of lesson design was common in other countries, or is it just unique to the UK. Stigler, Fernandez and Yoshida (1996) in a study of the similarities and differences in lesson design in the USA and Japan describe a typical lesson plan from the USA in terms of actions to be performed by the teacher and pupils:-

> Teacher reviews a mathematical concept
> Teacher explains a new related concept
> Pupils do practice examples
> Teacher explains an extension to the new concept
> Pupils do practice examples
> Pupils work individually on an exercise.

In Japan they found that lesson plans were written with more of a dialogue between teacher and pupil in mind and might typically include:-

Teacher presents a complex problem
Pupils attempt to solve the problem on their own or groups
Pupil explain their solutions
Class discussion of pupil solutions, combined with teacher explanations
Agreed general solution
Pupils work on practice problems

Lessons plans vary in complexity, detail and general structure or actions to be taken by the teacher or pupils (John, 1991, 1994). As John (1993) claims, “virtually all major guide books on curriculum and lesson planning begin with the importance of laying down, at an early stage, the educational and learning goals that will guide the lesson” (p. 30). This view reflects, and is supported by, what Barnes, Clarke and Stephens (2000) call a more “rational planning model”, which has its foundation in a better alignment between objectives, classroom practice, and evaluation and typically involves, specifying objectives, selecting and sequencing learning episodes and finally evaluating the outcomes of the learning episodes. This lesson design framework may have the effect of encouraging professional conversations and hence the development of a common pedagogical language albeit local defined between collaborating practitioners.

Suggested lesson planning templates would seem therefore to advocate framing learning in terms of teacher actions such as specifying objectives or aims or the presentation of a mathematical problems or content. Few lessons are defined in terms of learning or the processes that are planned to encourage learning to take place. Relatively fewer exist that aim to help teachers with the selection of the specifics of the types of learning episodes (such as activities, skills, exercises and tasks), but it could be argued that if adopted this would become as restrictive as that of the UK Key Stage 3 National Strategies model described above (DfEE, 2001).

The themes and principles are at the core of designing learning episodes as described by Ainley, Pratt and Hansen (2006), whilst investigating hard to teach topics in mathematics. Topics, such as the manipulation of fractions, are explored through the similarities and differences between the interchangeable uses of professional terminology. The following demonstrates the very essence of the argument concerning the interchangeable usage of terms when describing lesson mathematical content (bold underlining emphasis is mine)

> When manipulative **<u>activities</u>** are utilized, they should provide a framework for understanding (Baroody, 1989), but manipulative **<u>activities</u>** should not interfere with the teaching of the content. Because **<u>tasks</u>** with manipulatives can be very time-consuming, the **<u>activities</u>** should be carefully planned where **<u>skills practice</u>** and review are important (Carnine, Jitendra and Silbert, 1997, p. 69)

Here the above authors are warning us about the need for a good reason to use manipulatives and the impacts on planning and lesson time. They are differentiating between the terms task, activity and skills and implying that a task may be dependent on activities and skills. Bouck and Park (2018) found that the use of a manipulatives in mathematics lessons to be an unusual practice. Whilst working with a group of trainee teachers Santagata, Zannoni and Stigler (2007, p. 125) argued that “a shared language to describe innovative and effective teaching practices are lacking” but physical objects (manipulatives) help to begin the process of defining a common shared language.

Therefore the essence of the argument behind this study is that there are differences in meanings and enactment of the pedagogical words task, exercise, skill and activity when designing learning episodes. Skills and exercises are often synonymous in mathematics, at school level, with the practice questions found in textbooks or worksheets, whereas tasks or activities are more likely to be associated with open-end problems or investigations. If teachers are to support conceptual understanding we need a precise language for lesson planning which clearly defines learning terms and allow us to distinguish and promote the most appropriate learning experiences for our pupils.

2.11 How the literature has contributed to my thinking?

Since I started an in depth review of the literature for this study there has been a number of critical moments and turning points in my thinking. For example, the social constructivist, situated- and problem-based learning literature had raised my awareness of the alternatives to the approaches I had been schooled in as a trainee teacher in the 1970s. Whilst I was aware of different learning theories the reading confirmed my suspicions that the traditional didactic approach to the teaching of mathematics was not the only model and that my own beliefs and values aligned more closely with the other learning theories such as social constructivism and experiential.

The mathematical thinking literature helped me to see mathematics clearly as a thinking process rather than a body of knowledge. Additionally the literature concerning different aspects or phases of a mathematics lesson led me to believe that a standard mathematics lesson where a teacher demonstrates a worked example followed by pupils working on a textbook exercise of similar questions was not the only way to learn mathematics. It was not until I had read Ainley's (2006) paper "Task Design based on *Purpose* and *Utility*" that I fully realised that mathematics is more than the acquisition of knowledge and algorithms and I began to picture how designing mathematics lesson might be done differently.

Overall, the literature relating to the four selected aspects of a lesson (activity, skill, exercise and task) reinforced my early beliefs that these were often used interchangeably. However having synthesised the literature I was able to distinguish features for each of the four aspects and this allowed me to define working definitions to use during the study and to share with the participating teachers (see the table at the end of this chapter).

Therefore, this body of literature had shaped my thinking and provided me with direction and focus for my own research but there did not seem to be any single case study or piece of research that tackled the range and combination of issues that I needed to address. Most of the research focused on one aspect of a lesson design such as a skill or a task. I would be instigating and observing several changes and their effects on a specific group of pupils within a specific learning situation (division of fractions). Although I would not be able to make generalisations from a single research situation, I would be able to provide a rich picture of the aspects of lesson design and the impact on pupil learning. I believed this detailed study in one school setting would be useful as a resource for practitioners seeking to change their practice. In the following section I consider how my investigation might contribute to current research.

2.12 How will my research study contribute to current knowledge?

If successful, this research would make a valuable contribution to the understanding of the differences between the four types of learning episodes (activity, skill, exercise and task) which teachers often use interchangeably. The effects of an alternative lesson design and the use of a non-standard approach to the teaching the division of fractions will develop mathematical teaching approaches and inform practising teachers.

I had come across very few studies that evaluated the effects of this sort of change in the lesson design and the impact on pupil learning and achievement. There had, however, been numerous studies into effective ways of teaching the division of fractions (Kieren, 1976, 1980, 1988, 1993; Vergnaud, 1983; Behr et al., 1993; Small, 2009) but all of these studies were mainly interested in the change of the mathematical approach rather than the lesson design.

I wanted to find out if change of lesson design using one of the non-standard but already researched approaches to the teaching of fractions would result in children of all achievements being able to access a difficult mathematical concept. I hoped my evaluation of a change in lesson design would contribute to the research into developing and enhancing lesson design and informing practitioners.

By influencing and exploring the relationship between lesson design, mathematics and pupil achievement I had planned to encourage teaching professionals to use these alternative strategies. I also wanted to identify if a change in lesson design might move the practice of other professionals towards designing lessons with a clear view of the definitions of the four learning episodes. I was not aware of any other study that set out to study these relationships. I not only set out to do this, I had also set out to influence the changes in other professionals especially as I am currently a teacher educator. I was aware that it would be difficult to make sense of such complex relationships and be able to identify cause and effect. Perhaps that is why there are so few studies of this nature. However I was keen to explore the whole situation of lesson design, a change in teaching approach from a standard algorithmic approach and professional development of teachers. I felt, therefore, that a case study would shed light on the interrelatedness of issues involved in such a change.

2.13 Summary and conclusion

This review of the literature has helped me address the two research questions that had begun as an interest into why all mathematics lessons feel and look the same in both style and design. The link between professional language terminology as a component of lesson design features and the mathematics shared with pupils is poorly understood, but given that lesson design has a relatively recent research history this is not surprising. The interchangeable nature of terminology does not make for clarity of understanding even though most teachers would admit they profess to understand what colleagues are expressing. The professional vocabulary that teachers use to communicate with each other about learning and teaching is often fuzzy but paradoxically generally well understood by those working in the profession.

This review of the literature did help me with highlighting the interconnections between the research questions but more significantly the literature did confirm my view that a clear shared pedagogical language for lesson design is of importance for both the teacher and the learner. The literature also gave me a deeper insight into the number of ways in which the division of fractions is taught and confirmed my view that some 'standard' ways of teaching particular topics are not necessarily the most cognitively effective for the learner. The literature also raised the question of how else mathematics could be planned and taught. It also made me consider how the pupil outcomes from a change in design of a lesson might be measured and how the change in a lesson design might influence participating teachers. The social constructivist, situated- and problem-based learning literature enabled me to consider alternative approaches to teaching and pointed me towards a way of conceptualising the different aspects of the pedagogical terms under investigation.
Certainly the literature helped me to formulate working definitions for the four terms (activity, skill, exercise and task) which I consider to be fundamental learning episodes when designing a mathematics lesson. In this respect the

literature around pedagogical language was helpful in identifying the problems and care in the use of language to design lessons that I would need to take when talking with teachers in the participating school.

Trying to synthesise exactly the individual characteristics of each of the four pedagogical terms (activity, skill, exercise and task) led me to the belief that there are inevitable overlaps in their meanings. The review of the literature did point towards the fact that one of many reasons for studying mathematics is for learners to acquire problem solving expertise. With the recent debate about fluency and mastery in mathematics (Foster, 2017; Howard, 2018) being essential to mathematical development then, these four terms are inevitably intimately bound to the ways in which the subject is presented to learners. The purpose of defining these four pedagogical terms was to promote a structure for the way in which the subject is presented by teachers to encourage competent, fluent mathematical learners. In this structure a task is seen to be where fluency and mastery are demonstrated with the role of a skill being to present new knowledge, an exercise as an opportunity for practise and a degree of self-exploration of the subject; and an activity as a means of encouraging knowledge recall. The definition of a task is heavily reliant on the work of Ainley's (2008) view of purpose and utility together with Swan's (2005) views of richness and collaboration.

I acknowledge that the separations in meanings of the four pedagogical terms as defined early are neither watertight, nor exclusive conceptual categories. However these have been developed in the interests of trying to improve the pedagogical language of mathematics teachers with the aim of coming to a common shared understanding. There is inevitably some conceptual fluidity between aspects which I recognise and that others may question in the definitions proposed.

The definition of an activity, presented here, as not introducing any new learning is slightly problematic. Social constructivist theory tells us that learning takes place when learners interact, and this is a fundamental aspect of an activity, hence learning is likely to occur. The difference between the definition of a skill and that of an exercise lies in pupils posing their own questions when engaged with an aspect of learning defined as an exercise. The definition of a skill has the development of mathematical fluency at its core (Taleporos, 2005; Ofsted, 2012; Foster, 2013). Contrastingly the definition of an exercise promotes the opportunity for pupils to demonstrate mastery (Hewitt, 2015; Foster 2017). Whilst the distinctions between the four terms are subtle, with inevitable overlaps, this just serves to exemplify the problems teachers have when using pedagogical language. Trying to clearly and unambiguously define commonly understood pedagogical terms in some respects replicates the view that learning is a “very complex matter, and there is no generally accepted definition of the concept” (Illeris, 2018, p. 1).

Trying to establish an optimum order in which these four pedagogical terms should be used to structure a lesson is not part of this research, but it would seem logical to start a lesson with an activity to recall prior learning. The order in which the other three are used is open to debate and further research, however, I decided to use the order activity, skill, exercise and task for the study lesson in this research.

Finally for completeness and as a recap I will be using the definitions for the four pedagogical terms under investigation in this study as detailed in appendix 35. These were shared with the teachers involved in the study.

Chapter 3 – Methodology and Design

3.1 Introduction

Philosophical assumptions relating to what constitutes valid research and the appropriate selection of research methods nearly always underpin the development of new knowledge (Khalid, 2017). It is important for clarity, transparency and ethical obligations to explicitly state what assumptions are made during the research. The purpose of this chapter is therefore to discuss a research methodology and the research tools used in the study and as a result this chapter is in two sections. The first section presents the methodology for the study; additionally it suggests how this study supports previous studies into collaborative research and student teachers' beliefs. This section also considers the advantages and disadvantages of using qualitative case study research as the methodology for this study whilst also discussing reliability, validity and ethical issues. The second section describes the research instruments and methods to be used in the data analysis. The design of each research instrument is discussed and how the data would be collected and analysed.

I absolutely agree with Burton (2005) that researchers often explain ***how*** results are arrived at (methods) rather than ***why*** choices and decisions are taken which then inform research conclusions (methodology). According to Burton (2005) there is never a case "where the researcher's beliefs, attitudes, and values have not influenced a study" (p. 3). She also warns that a researcher should never "assume that values can be assumed as shared within a 'scientific community'" (ibid, p. 3). With these two stark warnings in mind I intend to clearly express my values and beliefs, which are the driving force behind this research, and hence make visible to the reader, my decisions on the selection of a particular methodology and methods employed. Research is predicated on underlying philosophical assumptions about what represents legitimate research together with which approaches and methods are appropriate for the development of knowledge. This chapter discusses the research methodologies, the design of this study including strategies, instruments, and the data collection and analysis methods to be used.

The research design for this study is descriptive, interpretative and based on a qualitative case study research methodology in a single school setting with two groups of 11-12 year old children and their mathematics teachers. An exploration of practice is used together with a descriptive statistics to analyse a particular aspect of pedagogical practice. Analysis of classroom videos, lesson plans, single and group teacher interviews and a survey of pupil views in the participating school together with a trainee teacher questionnaire survey are used as the data collection tools. In order to confirm trustworthiness of the research several methods appropriate to qualitative research are used and discussed.

3.2 The Drivers for this Research

Understanding how pedagogical approaches are used in the teaching of division of fractions and the influences of lesson designs are the central themes of this research. As stated in chapter 1, the two main research questions are

> **What are the influences of lesson design on pupil learning?**
>
> **What are the implications of a change of approach to teaching fractions for teacher training?**

with an additional two subsidiary research aims:

> **What types of learning episodes could support better pupil understanding of mathematics?**

and

> **What apparent mathematical misconceptions and barriers prevent pupil progress?**

My conjectures at the outset of the research were that the types of work given to pupils and the teaching approaches taken by teachers have an influence on learning. I also conjectured that good subject knowledge and receptiveness to a change of mathematical approach by teachers might also influence pupil learning. I wondered, therefore, what might be the effect of a change of mathematical teaching approach on

(a) a teacher’s design of a lesson
(b) pupil learning.

It is inevitable that some of my beliefs about the impact of lesson design and terminology on learning have been formed and reformed during thirty years of secondary school mathematics teaching. Recently eight years of working as an initial teacher educator have also further impacted on my beliefs. Trainee teachers, in my experience, often do not equate lesson design with learning as they tend to concentrate on the factual content or mathematical processes. The design for this study is based on a descriptive, interpretive, participatory case study in a single setting that then analyses the data through qualitative methods. Questionnaires were used to evaluate trainee teachers' beliefs and views before and during the research. Descriptive statistical methods were used to analyse the questionnaires. Teacher participants were involved in the classroom fieldwork, face-to-face semi-structured interviews, and video recordings of lessons were used as data collection tools. Pupils were obviously involved, but not as active researchers, however their views were sought to enrich the narrative (section 3.3.2).

My ontological and epistemological assumptions, and my views concerning learning, teaching and mathematics, will impact directly on my selection of an appropriate research methodology. Cohen et al. (2013, p. 8) claim that how you view the world will impact on "the choice of problem; the formulation of questions to be answered; the characterisation of pupils and teachers; methodological concerns; and the kinds of data sought and their mode of treatment". A researcher's epistemological assumptions arise from their beliefs where a positivist view is questioned by Cohen et al. (2013, p. 11) because of the "complexity of human nature and the elusive and intangible quality of social phenomena". Hammersley (2002) suggests that adopting a positivist epistemological stance is "inappropriate in educational investigations" as there is a "values dimension" in education research that a scientific model fails to address (Burton and Bartlett, 2005, p. 5). Given my positivist mathematical background my ontological beliefs of the social world and education have led me to an interpretive, qualitative approach to this educational research. Using an interpretive paradigm will still aim to apply the same rigour as the natural sciences whilst being concerned with explaining human behaviour to emphasise "how people differ from inanimate natural phenomena and indeed, from each other" (Cohen et al., 2007, p. 7).

My two research questions led me to consider both quantitative and qualitative methodologies. Bryman (2008) asserts that the selection of a specific methodology should be firmly linked to the research questions. Denzin and Lincoln (2011) also remind us that qualitative research emphasises discovery of social meaning and stresses the relationship between the researcher and the topic studied. Aliaga and Gunderson (2002, in Muijs, 2004, p. 1) define quantitative research as the exploration of a “phenomenon by collecting numerical data that are analysed using mathematically based methods”. Quantitative research, by contrast, tends to investigate causal relationships between variables by taking an empirical stance through measurements. Reflecting on the two research questions it was considered to be extremely unlikely that numerical quantities would form a huge part of the research. This would be because of the very nature of the research questions and the need for a more descriptive, in-depth approach rather than a measuring of variables. A number of quantitative studies into teachers’ beliefs (Ball, 1992; Nisbet and Warren, 2000; Yates, 2006), using numerical data as a means of extracting teacher beliefs have been conducted. These studies have been found to be of limited use because they tend to describe what is happening rather why it is happening. A fuller, richer dataset can be achieved through mixed methods approach to gain a deeper understanding of what is happening and why.

As a former practising mathematics teacher with some working knowledge of research methodologies, I naturally gravitated towards a quantitative approach. So I searched for a methodology that would be both flexible and supportive whilst accommodating both qualitative and quantitative approaches. This, I felt, would allow for the gathering of rich data, with an increasing refinement of both insights and detail as the research progressed. As Major and Savin-Baden (2010, p. 14) reminds us “the two approaches taken together could provide powerful information”.

For Wright (1995) qualitative research means less reliance on number counting and statistical techniques in favour of getting closer to the data collected from natural settings. The emphasis of qualitative research is often to understand comprehensive, interdependent, holistic structures that are dynamic and predictive (Wright, 1995). Holistic structures tend to suggest that any individual variable or combinations of variables are important in the interaction, a view shared by Kleiner and Okeke (1991). For Van Maanen et al. (1982) a characteristic feature of qualitative research is the reliance on multiple sources of data, rather than just one source, so as to be able to explain events.

3.3 Research Design - A Qualitative Case Study

My intention was therefore to find a methodological approach that would give a degree of flexibility when investigating the two main research questions:-

1. **What are the influences of lesson design on pupil learning?**
2. **What are the implications of a change of approach to teaching fractions for teacher training?**

Having read extensively about different methodological approaches to research I had originally contemplated two research design methodologies (action research and case study) as being appropriate for this research. It soon became apparent that the research questions involved multiple facets and that classroom work would be needed. I would also have to be in classrooms working with other professionals and needed to engage in dialogue and gain the trust of others. A qualitative case study methodology would allow for this and it also it came to my notice that teachers in a partnership school, who were studying at master's level, wanted to be actively engaged in this study as a means of improving their knowledge, understanding and professional practice. As Yin (2013, p. 10) states

> case studies, like experiments, are generalizable to theoretical propositions and not to populations or universes. In this sense, the case study, like the experiment, does not represent a "sample", and the investigator's goal is to expand and generalize theories (analytic generalization) and not to enumerate frequencies (statistical generalization).

Whilst consulting the literature the term case study is often referred to, and used as, both a methodology and a method. Mills (2014) distinguishes methods as procedures and techniques employed in the study, while methodology is the lens through which the researcher views and makes decisions about the study. Given the variation in definitions and descriptions, referring to case study research as a methodology and/or a single method can be perplexing (Anthony and Jack, 2009; Flyvbjerg, 2011). Furthermore, proponents of case study encourage the use of both quantitative and qualitative methods which can add yet another layer of complication (Merriam, 1998; Stake, 1995; Yin, 2013; Stewart, 2014).

Stake (1995) argues that case study research is qualitative and closely aligned with a constructivist and interpretive paradigm so as to be able to discover meaning and understanding of experiences in a context. Taking an interpretative positionality allows the researcher to view reality as multiple and subjective, based on meanings and understanding. The knowledge that is generated from the research is therefore relative to the time and context of the study with the researcher is interactive and participates in the study. In terms of epistemology, Stake (1995) argues that the situation shapes the activity, experience, and one's interpretation of the case and "requires experiencing the activity of the case as it occurs in its context and in its particular situation" (p. 2). The researcher therefore attempts to capture an interpreted reality of the case, while studying the case in situ enables an in-depth examination of the system as the research unfolds.

Nevertheless, having made this decision to use an in-depth qualitative case study rather than action research which relies on multiple cyclic iterations of research, I soon realized it would allow for the added advantage of the teachers being present in the classroom and partially involved rather than fully participative. This was considered to be a real positive, for the collection of the data to be used in answering the research questions, the involvement of other professionals and putting pupils at ease in order to gain ‘real’ or ‘natural’ data. I felt this methodological approach would be the most appropriate giving me the authentic, informative views from all concerned in order to answer the research questions. Yin (2013) argues that use of a case study methodology is particularly suited to situations in which the researcher has relatively little control over events or the phenomenon under investigation. In complex situations, such as a classroom, where it is not possible to easily separate the boundaries of the phenomenon under investigation from the case being explored then a case study is a particularly suitable methodological approach.

This research, therefore, employs a qualitative single case study design with a group of participating mathematics teachers in one school (see section 1.3.5) which included both newly qualified and more experienced teachers. The table below shows the contribution each teacher made to the various parts of the research.

Teacher	Experience	Involved in classroom research	Survey	Lesson plans
A - Female	2 years	No	No	Yes
B - Male	4 years	No	Yes	Yes
C - Male	6 years	Yes	Yes	Yes
D - Female	NQT	Yes	Yes	Yes
E - Female	7 years(manager)	No	Yes	Yes
F - Male	6 years	No	Yes	No
G - Female	1 year	Yes	Yes	Yes
H - Female	3 years	Yes	Yes	No
I - Female	7 years (manager)	Yes	Yes	Yes
J - Female	10+ years (manager)	No	Yes	No
K - Female	8 years	No	No	No

Table 3.3 Teacher contributions to the research.

Table 3.3.1 (in the table section at the end of the thesis) contains a fuller commentary for each teacher. A qualitative case study is a type of research that makes use of an in-depth analysis of a bounded entity which could be a person, a place, or an event (Stacks, 2005). A case study is therefore a holistic inquiry that investigates a contemporary phenomenon within its natural setting, which usually relates to some relevant issue where researchers reveal information about some phenomena through the process of this detailed study (Putney, 2010). Denzin and Lincoln (2011) argue that qualitative research involves an assortment of methods and approaches which Flick (2014, p. 542) claims are needed for “qualitative research interested in analysing subjective meaning or the social production of issues, events, or practices”. This statement stresses how researchers make sense of something in the world. But originally Van Maanen (1979, p. 520) had defined qualitative research as, “an umbrella term covering an array of interpretive techniques which seek to describe, decode, translate, and otherwise come to terms with the meaning, not the frequency, of certain more or less naturally occurring phenomena in the social world”. With Prasad (2005) arguing for the co-existence of multiple paradigms under the broad umbrella of “qualitative research” it would therefore appear to be an all embracing theory which seeks to accommodate the variety and diversity of issues and phenomena. The flexibility inherent in adopting a qualitative research approach allowed me to probe some of the superficial data, a teacher thoughts and emotional responses.

This is critically important because often it is an emotional response which drives a person's decisions and influences their behaviour (Denzin and Lincoln, 2011).

3.3.1 Advantages and Disadvantages of Qualitative Research

A major advantage for this study of using qualitative research is that it can produce detailed descriptions of participants' feelings, opinions, and experiences; and moreover can interpret the meanings of their actions (Denzin and Lincoln, 2011). Additionally employing a qualitative research methodological approach (interpretivism) aims to holistically understand the human experience in a specific setting. Denzin and Lincoln (2011), for example, mentioned that qualitative research is an interdisciplinary field which encompasses a wider range of epistemological viewpoints, research methods, and interpretive techniques to understand human experiences. An interpretive research approach is also regarded as an ideographic research methodology in that it affords the researcher the opportunities to study individual cases, events and to understand different people's voices (Klein and Myers, 1999). Corbin and Strauss (2008) argue that qualitative research facilitates, for the researcher, the ability to reveal participants' inner experience, and how meanings might be shaped. A final advantage of a qualitative research methodology is that qualitative research methods such as participant-observation and participation, unstructured interviews, direct observation, are most commonly used for collecting data (Cohen et al., 2013). During the data collection for this type of study interactions with the participants (teachers and pupils) are viewed as a necessity. In this respect, a qualitative research approach was required to capture the dynamics of the problem being investigate and that this was an appropriate methodology (Mohan, 2012). It was therefore expected that qualitative research would contribute to the understanding of the complex features of lesson pedagogy.

Allied to advantages are obvious limitations or disadvantages with Silverman (2013) arguing that qualitative research approaches can sometimes leave out contextual sensitivities, and focus more on meanings and experiences. Investigating particular phenomena attempts to uncover and interpret a participant's experience (Wilson, 2014; Tuohy et al., 2013) rather than any other issues in the context. In terms of research method, a small sample size (one school) raises the issue of generalisability of the research (Harry and Lipsky, 2014; Thomson, 2011). However Lam (2015) argues that qualitative studies involving a single case context do not wish to claim wider generalization to other contexts. The value of using a case study approach for this research lies in what Yin (2013, p. 14) states is a problem can be explored “in depth and within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident”. Additional Stake (2008, p. 443) argues that a case study research is often “defined by interest in an individual case, not by the methods of inquiry used”, nor the wish to for generalisations to be made.

Berg and Lune (2012, p. 4) also commented that, “qualitative research is a long hard road, with elusive data on one side and stringent requirements for analysis on the other” with Flick (2011) pointing out that the analysis of a single case using qualitative methodology takes a considerable amount of time, and can only be generalised in very limited ways. Despite these perceived limitations qualitative research has become prominent in educational research (Manias and McNamara, 2015). Moreover, the generalisability seems not to be a problem as Darlington and Scott (2003, p. 118) infers stating that “if one considers the unit of attention as the phenomenon under investigation, rather than the number of individuals, then the sample is often much larger than first appears”. Finally Labaree (2004) suggested that no educational research (either quantitative or qualitative) ought to be regarded as generalisable, because too many contextual variables can shape the findings. With Donmoyer (2012) arguing that qualitative researchers highlighting to policy makers what works provided through thick description there does appear to be a rational for adopting this approach. The particular study I have designed is a qualitative case study because I am interested in developing a deeper understanding of the phenomenon of lesson design and in particular a teacher’s use of pedagogical terminology. I was also mindful of the findings of Starman et al. (2013) in the choice of a qualitative case study methodology with participating teachers as they had found “collaborative opportunities – internal and external to an institution – can provide new information and resources to enhance instruction” (p. 85) and more importantly the “new information and resources facilitates improvement in student engagement and learning, which can lead to improved student outcomes” (p. 86).

3.3.2 The Rationale for Case Study Methodology

For Yin (2013) a case study is an empirical inquiry which focuses on a contemporary phenomenon within its real-life context and boundaries between phenomenon and its context are not clearly evident. A case study is a suitable approach for studying complex social phenomena with many variables and multiple sources of evidence with both qualitative and quantitative data collection tools. So, Yin (2013) holds the view that a case study is an empirical enquiry which investigates a contemporary phenomenon in a real life context. Basit further refines this viewpoint by adding that

> A case study provides a unique portrayal of real people in a real social situation by means of vivid accounts of events, feelings and perceptions (2010, p. 19).

According to McDonald and Walker (1975, p. 2) a case study is “the examination of an instance in action”; where the use of the word instance is important and deliberate in that it suggests the findings can be generalised; however; a contrasting definition from Stake (1995, p. xi) of a case study as “the particularity and complexity of a single case, coming to understand its activity”. It is this later definition that this research has used as the underlying methodology. Stake (1995) goes on to identify three board types of case study (Instrumental, collective and intrinsic) and Bassey (1999) adds a further five (theory-seeking, theory testing, story-telling and picture drawing and evaluative). Yin (2013) identifies five categories (explanatory, descriptive, illustrative, exploratory and meta-evaluation) and argues that explanatory is the most important “to explain the causal links in real-life interventions that are too complex to survey” (p. 15). These broad categories of case serve to exemplify the complexities and problems for the researcher when using case–based methodology as a basis for social science research. Nevertheless this research was grounded in an explanatory case study approach.

Using a case study methodology to investigate complex phenomena invites the risk of ill-defined or poorly conceived approaches. The definition of the case or object under study and the issues arising from the limits or what Carter and Sealey (2009, p. 69) call “boundedness” of the study are serious ontological and epistemological questions for the researcher. Simons (2009) argues that bounding the case whilst a good idea “may change once you enter the field” (p. 29); and that a single case might be a class or an institution and the boundaries include people, politics, policies, location, to which I would add pedagogy.

Boundaries might need to be refocused and can be somewhat fluid and responsive to outcomes of the fieldwork; however with boundary fluidity comes dangers such as lack of compactness and conciseness.

In contrast Ragin (1992) argues that researchers should be absolutely rigorous and explicit about the limits and processes framing the case under study in order to extract clear meanings from the work. Identification of the domain and processes further frames the case and provides realism for what can and should be included. Case study is therefore both flexible and time independent and is not constrained by the selected methods (Simons, 2009, p. 23) but it does provide the opportunity for a "self-reflexive approach to understand the case and themselves [researcher]" (p. 23).

Whereas objects or phenomena from natural sciences are susceptible to case methodology as a means of providing "useful information about similar objects [cases]" Carter and Sealey (2009, p. 70) argue that when applied to social sciences it inevitably involves a degree of reflectivity. Archer (2003) argues that this reflective nature of the research involves an "active process in which we continuously converse with ourselves, precisely in order to define what we do believe, do desire and intend to do" (p. 34). Explicitly identifying your actions, values, beliefs, preferences and biases that could potentially influence the research process enables others to see how the interpretations and conclusions are achieved. Probably more importantly it allows the researcher and others the opportunity to detect potential bias in the study.

Using participant teachers for the case study not only provides "opportunities for collecting case-study data, but also it provides major problems" (Yin, 2013, p. 112). The evidence collected from this approach can be "insightful into interpersonal behaviour and motives" (ibid, p. 162) but has to be counter-balanced to prevent "bias due to participant-observer's manipulations" (ibid, p. 162). Potential biases of the researcher to steer or guide the study and the tension between the ability to act as an observer as well as a participant should not be underestimated. The prospect of being able to gain access to situations [classrooms], events [lessons] and group [teachers and pupils] is fundamental to this researcher and I would argue the only way of achieving the evidence needed.

More generally Cohen et al. (2013, p. 181) explore the notion that a case study is "a specific instance that is frequently designed to illustrate a more general principle". They add that a case study "provides a unique example of real people in real situations" (p. 181). The concept of generalisation from a case study is supported by Robson (2002) who argues that case studies offer analytical generalisations. This might be seen as supporting the viewpoint that tasks designed on a generic set of underlying constructs in a particular context (division of fractions) can be generalised to other contexts. However Pring (2004, p. 40) argued that "the uniqueness of events or actions" points to the case study as a "unique case or instance" and therefore care needs to be taken when inferring generalisation based on a single context. Pedrosa et al. (2012) suggest that case study research should be evaluated therefore not only on results (validity and reliability) but also on the entire research and importantly the process should be transparent to the audience to achieve validity and reliability.

3.3.3 Reflexivity and Power Relationships

Qualitative researchers need to guard against, or at the very least recognise and acknowledge that, their ontological and epistemological beliefs impact on their research. These beliefs also impact on their positionality. Ramani, Könings, and Mann (2018, p. 1257) argue that "qualitative researchers must engage in reflexivity—at all stages of the qualitative research process. They must recognize their beliefs and assumptions, acknowledge their relationship to the research topic and participants, and consider how these influence their study". Carducci et al. (2013, p. 8) states that "issues of reflexivity and positionality are messy and yet we often paint them as not so". However, it is generally argued that reflexivity strengthens research and is indeed a methodological imperative in qualitative inquiry (Salzman, 2002; Lather, 2003; Richardson, 2005; Jones, Torres and Arminio, 2006) to challenge researchers to explicitly articulate the influences on their research.

Luttrell (2010) argues that reflexivity is a prized tool for the qualitative researcher and should not be confused with being reflective, the distinction being "reflection is a means of looking back or looking more deeply to gain insight, but reflexivity is a process of awareness and scrutiny" (p. 14). Archer (2012) also reminds us that reflexivity is a form of internal dialogue that researchers undertake to lead to actions which can transform practices in the classroom. A researcher who adopts a reflexive approach to classroom research “can promote deep professional learning and bring sustainable change in education” (Feucht et al., 2017, p. 234) and it involves critical thinking to evaluate “multiple perspectives [which] leads to action in the classroom” (ibid, p. 238). Pintrich (2002) identifies personal epistemologies, beliefs and cognition about knowledge as key factors in reflexivity. The implication here is that internal dialogues associated with reflexivity needs to include reference to both teaching practices and epistemic cognition as a process so that it can lead to action in the classroom (Archer, 2012). Reflexivity is therefore a tool where we can include ourselves in the research as long as we make “transparent the values and beliefs we hold that almost certainly influence the research process and its outcomes” and allows us to demonstrate for the reader “*what* we have discovered, but *how* we have discovered it” (Etherington, 2007, p. 601).

Gewirtz and Cribb (2006) had set out a clear set of criteria for ethical reflexivity which involves being explicit about the values, assumptions and evaluative judgements that inform each stage of the research. I was aware that my ontological positioning would require a high degree of self-reflexivity and I anticipated some of ethical dilemmas that might arise from my relationships with the teachers, pupils and the study school due to an ‘insider’ (a mathematics teacher), and an ‘outsider’ (university lecturer external to my own workplace) roles. I was also drawn to what Ari and Enosh (2012) highlight as the knowledge is co-constructed through the process of reflexivity, as participants exert power in shaping knowledge through choosing what wish to reveal. Weinstock et al. (2017) argue that reflexivity is sometimes implicit and often an ambition of learning environments and of the curricula. Moreover they argue that reflexive thinking is the type of reasoning that promotes knowledge cognition and informed reflexive thinking is an epistemic virtue that supports and underpins the research process.

Thinking about the power relationships between the researcher and ex-student teachers I had initially considered this to be an equal relationship, but I was mindful of Merriam et al. (2010, p. 408) who advised not to overlook “the multidimensional power relationship shaped by the prevailing cultural values, gender, educational background and seniority". Often there is an asymmetry of power between interviewer and interviewee when conducting research with ex-students, with the balance residing with the interviewer. It is evident that the interviewer holds a powerful position after collecting confidential information after the interviews; however the interviewee holds the ultimate power in that they can refuse to continue (Powney and Watts, 2018).

As Hamzeta et al. (2018) explain there is a dichotomy when trying to alleviate power relationships and this can have unintended risks for the research. Researchers adopting their traditional expert role as experts whilst teachers assume less responsibility or risk can be problematic. This was not the approach adopted here; collaboration and negotiation were foremost in my mind when dealing with the teachers. The stance I adopted was both friendly and relaxed; continually reminding the teachers that I valued their honest contributions. However I openly, and fully acknowledge, that even given the degree of care taken some of the power relationship could possibly still have been present.

3.3.4 Issues of validity, reliability and ethics

A central tenet of all research irrespective of the methodological approach is that of our ability to learn and develop new knowledge with integrity, responsibly and ethically (O'Leary, 2017). It is therefore essential that research communities have confidence in the conduct of the research activities and the results of investigations: this is particularly important for this study due to the applied nature of the educational inquiry (Merriam, 1998). Issues relating to research findings are intimately associated with seeking the 'truth' (validity), their generalizability (reliability), and the impact on the participants (ethical concerns). This section therefore discusses the validity, reliability and ethical issues that relate to this study.

3.3.5 Validity

Validity for Ritchie and Lewis (2003) lies in the correctness or precision of the research findings. Basit (2010, p. 64) reminds us that "no research is totally valid as threats to validity cannot be totally removed" but "validity is a vital element of effective research because if a particular study is invalid, then it is worthless" (ibid, 2010, p. 64). Validity as applied to qualitative research findings is often perceived through the challenge and defence of believable data whilst simultaneously considering whether the data is plausible, credible and reliable. Contrastingly, validity is frequently achieved in quantitative research through the use of mathematics or statistics where results are often definite and provable in nature. Qualitative research validity tends to be less statistical and relies more on data analysis that produces rich, deep narratives. Some qualitative researchers believe that the concept of validity is not synonymous to validity as understood by quantitative researchers and therefore a different perspective or approach should be employed. Those researchers arguing from this standpoint still make the case that every effort should be made to ensure validity so that the results of qualitative research are to be believed. Other researchers do not make any distinction between achieving validity via different methodological approaches, for example Hammersley (1987, p. 69) argues that “an account is valid or true if it represents accurately those features of the phenomena, that it is intended to describe, explain or theorise”.

As an initial teacher trainer of the newly qualified teachers, some of whom were selected for involvement in this research, I openly acknowledge that I may have some preconceptions, for example about participant’s mathematical subject knowledge and even value judgements about their pedagogical approaches. Consequently the potential for bias is acknowledged and an open, honest, sharing approach already employed during their initial teacher training was continued to help guard against such influences. Shipman (1997) argues that total objectivity in social research may not be achievable, but by providing robust evidence trails of the research methodology, and also of the procedures and processes that underpin the analysis, then a degree of rigour can be achieved. In an effort to alleviate bias I needed to consider an open, honest and transparent approach in this study which could easily be evidenced through the design, discussions and research with colleagues.

Remarkably Lather (1986) and Morley (1991) argue for the rejection of researcher neutrality, as most research is often undertaken by stakeholders with an interest in resolving a problem. They also argue that neither qualitative nor quantitative methodologies are sufficient epistemological structures to guard against bias. A praxis paradigm where knowledge is derived from practice, and practice informed by knowledge, in an ongoing process, is a basis of research and this might be an appropriate approach for openness, honesty and transparency. I agree with Lather (1986) and Morley (1991) that total neutrality and total impartiality are almost impossible to achieve, especially where researchers have both a professional and emotional involvement with the participants, never-the-less they should be goals that are to be achieved by the researcher.

I further acknowledge that demonstrating credibility and validity when carrying out research in a single school and with a single group of teachers that I had trained would be an important consideration. I would further argue that external validity as a cause-effect relationship between the independent and dependent variables is more normally applied to quantitative research, whereas internal credibility as a means of demonstrating believability is more appropriate for my qualitative study. However, Greenwood and Levin (2006, p. 80) argue that knowledge from research has an "*internal credibility"* when generated collaboratively, but equally they warn that *external credibility* is required to convince "someone who did not participate in the inquiry that the results are believable" (ibid, p. 81). Such internal credibility, I would argue, emerges and is created through reflection and discourse when "participants and the researchers negotiate the meanings created by their experiences during the research process" (ibid, p. 114). The nature of this qualitative case study is well suited to discourse and reflection from all involved. External credibility leads to Guba and Lincoln's (1984) view that for the findings to be trustworthy they should also be transferable and applicable in other contexts.

Therefore, in this study I take the view of Lecompte and Goets (1982) that the two forms of validity most applicable to qualitative research are

1 internal validity, the degree of connection between observed investigations and the theoretical ideas

and

2 external validity as the amount of generalizability of the findings.

3.3.5.1 Internal validity

A researcher's views, bias and values have an impact on the whole research process and they need to be acknowledged and carefully considered. It is also important, as a researcher, to declare professional experiences and preconceptions of classrooms and the relationships with other participants that may influence classroom interpretations and consequently the internal validity of the research. Such experiences and preconceptions were outlined in chapter 1. However, I would argue that my professional experiences, classroom conceptions together with strong professional relationships enabled me to better understand the classroom actions that I observe and allow me to interpret the views of participating teachers. Research needs to reflect, capture and critically analyse the perceptions of participants as "one of the assumptions of qualitative research is that reality is holistic, multidimensional, and ever-changing" (Merriam, 1998, p. 202). Several factors helped to increase the internal validity of this study including:

- long-term observations and a data gathering phase;
- the use of a participatory research model
- and, the triangulation of multiple methods of data collection as a method to confirm the findings

(Merriam, 1998).

Johnson (1997) and Newman and Benz (1998) support these three strategies as a means of demonstrating internal validity and suggest that researchers should also consider reflexivity (reflecting on ourselves and our involvement in the research. This is what May (2001, p. 44) calls "a consideration of the practice of research, our place within it" as the bidirectional relationship between cause and effect as well as to guard against their values creeping into the interpretation of data. Researchers should critically examine their interpretations to detect any potential bias which may influence the conclusions made about the data. Whilst it is recognised and acknowledged that conclusions can never be a hundred percent value-free, a researcher should strive to achieve high levels of honesty, transparency and openness in order to convince others of the merit of their study. The keeping of an audit trail consisting of accurate and fully documented data records allows others to have a degree of confidence in the evidence, results, and conclusions thereby providing an opportunity for challenge as well as validation of the interpretation of data.

My study can be said to be valid because I gathered data from different sources using different methods over an extended period of time. I planned to fully and accurately document the data gathered and record the date of all interactions. However, a weakness might be perceived in that with only one school setting used to collect the classroom based data, but two different classrooms were sampled. In terms of internal validity, a real strength of the study comes from the participation by professionals in the fieldwork, and their intimate knowledge of the subject material, the pupils and their full engagement during the whole process. Carrying out a range of checks to see if what I observe and record, equates to what the teachers observe and believe, is a real strength in demonstrating internal validity, especially given the participatory nature of the professionals involved. Taking data from multiple sources using a range of methods such as questionnaires, classroom observations, interviews, teacher lesson plans and outcomes from specifically designed teaching materials has both grounded and helped triangulate the results.

3.3.5.2 External validity

As stated earlier the external validity of the research is concerned with the generalizability of the findings from one context to another (Merriam, 1998). While notions of external validity in qualitative research are debatable, I would argue from my experiences that it is possible to enhance the generalizability of a study. Guba and Lincoln (1994) suggest two key criteria for assessing external validity in qualitative studies, credibility and transferability. Credibility viewed in terms of whether the findings are believable and transferability if the outcomes can be applied to other contexts. Silverman (2013) claims that external validity in qualitative research can be achieved through triangulation by comparing different kinds of data from quantitative and qualitative methods (questionnaires and classroom observations) to ascertain if they substantiate each other. Newman and Benz (1998) argue that findings can be externally verified if the study results are applied to other studies, or if in-depth description enables the researcher to generalise findings. Lincoln and Guba (1985) warn qualitative researchers to guard against making claims that the study can be generalised or transferred as this is not a necessity, these claims should be left to the reader. This study takes the combined views of Guba and Lincoln (1994) and Silverman (2003) and aims to achieve external validity by means of triangulation to demonstrate transferability and believability.

3.3.6 Reliability

Lincoln and Guba (1985) remind us that for qualitative researchers issues of reliability relate to dependability or consistency. Lincoln and Guba (1985) imply that research results need to make sense to outsiders and that looking at the same datasets would allow them to arrive at the same conclusions. In this study, reliability is increased by making explicit the theoretical underpinnings of the study, how the researcher is positioned, who the participants are and how they are selected, the context in which the data was collected, and how the triangulation of data was achieved (Merriam, 1998).

Reliability has its roots in quantitative research as a concept for testing or the evaluation of the research, however, this idea is now often applied to all research paradigms. The single most important assessment of a qualitative study is quality. Eisner (1991, p. 58) reminds us that reliable qualitative studies help us “understand a situation that would otherwise be enigmatic or confusing” with the intention of explaining and generating understanding. The difference between reliability as a testing mechanism in quantitative research and as a means of generating understanding leads Stenbacka (2001, p. 552) to argue that “the concept of reliability is even misleading in qualitative research”. Patton (2001) takes the opposing viewpoint reminding researchers that reliability as a measure of quality is a factor that should be carefully considered when designing any study and analysing the results. Joppe (2000) defines reliability as:

> The extent to which results are consistent over time and an accurate representation of the total population under study is referred to as reliability and if the results of a study can be reproduced under a similar methodology, then the research instrument is considered to be reliable (p. 1) .

Different paradigms should judge quality by their own terms (Healy and Perry, 2000). Lincoln and Guba (1985) suggest that reliability as a fundamental prerequisite for quality and closely correspond to credibility, dependability or transferability in qualitative paradigms. Clont (1992) and Seale (1999) support the concept of dependability as a means of defining reliability and Lincoln and Guba (1985, p. 316) state that: "Since there can be no validity without reliability, a demonstration of the former [validity] is sufficient to establish the latter [reliability]". Therefore, according to Guba and Lincoln (1985), the demonstration of a study being reliable through the use of appropriate methods, such as triangulation, may help to ensure or even guarantee that the research is valid.

The use of triangulation as a concept is justified as a means of confirming, demonstrating completeness and indicating the trustworthiness of data sets (Denzin, 1973; Jick, 1983; Bryman, 2008). Approaching research questions from differing viewpoints, using a range of methods, creates a deeper, fuller understanding as well as alternate interpretations. The use of triangulation is therefore defensible as a concept and as a method to underpin the exploration of the complex situations, as in this study, to check for validity and reliability (Bryman, 2008). Whilst triangulation is often used to support reliability, for consistent deep understanding we do need to be mindful of the temptation “to make inconsistent data sets artificially compatible in order to produce a more coherent account” (Arksey and Knight, 1999, p. 25). Triangulation cannot therefore be seen and a panacea to ensure reliability as it is possible to manipulate the data in such a way to achieve the result required.

The inherent “risk”, as an experienced professional, of prejudging the outcomes whilst undertaking the role of a researcher needed to be carefully thought through. Eliminating all risk is probably not feasible, but piloting all data instruments that were to be used in the research, is seen as an important step in the thinking and reflective processes and as a risk avoidance measure.

3.3.7 Triangulation

The concept of triangulation in research refers to the use of multiple, different approaches to generate a better understanding of a particular theory or phenomenon that is being studied (Burton and Obel, 2011). Fusch and Ness (2015) argue that “the application of triangulation (multiple sources of data) will go a long way towards enhancing the reliability of results” (p. 1411). Triangulation of data is therefore a critical factor for the reliability, trustworthiness, transparency, decision making and assumptions that are made (O’Brien et al., 2014; Arriaza et al., 2015), and this includes how the interpretations of data have been checked, cross-checked and any inferences that are made (Avenier and Thomas, 2015). Researchers using a qualitative methodology often use multiple data collection methods (triangulation) as it is assumed “that the use of a single method can never adequately shed light on a phenomenon” (Abrar et al., 2017, p. 17). Creswell (2009) reminds us that triangulation is also an important technique for a researcher when trying to ensure reliability and validity of the data. From a qualitative methodological standpoint data collection and analysis is unavoidably subjective in nature. So to enhance credibility and trustworthiness of the analytical processes when identifying meaningful interpretations of patterns “located in the subjective interpretation of data” (Levitt, 2015, p. 456) triangulation is a necessity.

The use of triangulation by methods such as interviews, lesson video observation, questionnaires and inspection of artefacts were all employed in this research so as to achieve a “richer, deeper, more robust, and also more well-developed data and to strengthen [their] validity and reliability” (Abrar et al., 2017, p. 19). This approach was important as Santiago-Delefosse et al. (2016) argue that data from different participants and at different times as well as using multiple data sources enhances reliability.

Therefore a primary purpose of using triangulation in this study was to eliminate or reduce biases and increase the reliability and validity of the data collected for the study. However, I also wanted to increase the comprehensiveness and confidence in the data to provide richness which would help with the understanding of the phenomenon under study. Twining et al. (2017) remind us that giving participants the opportunity to comment and correct interview manuscripts is also a valuable method of achieving credibility and trustworthiness with O'Brien et al. (2014) adding that emerging findings should also be shared for checking. Both of these suggestions were used in this research. The concept and use of triangulation of the data through the use multiple data collection methods enriches this study and the findings. This research therefore used two forms of triangulation – methodological triangulation (multiple methods) and respondent triangulation (different types of participants).

3.3.8 Ethics

Ethics in this research involved far more than a mere compliance with ethical codes and guidelines. There is an increasing recognition of the need to view ethics as a process throughout the entire life span of a project (Cutcliffe and Ramcharan, 2002; Guillemin and Gillam, 2004) rather than the process of mere compliance once codes have been agreed.

Groundwater-Smith and Mockler (2007) suggests that a set of ethical guidelines for this type of project should involve more than a mere observation of ethical codes and processes but include:

1. transparency and accountability to the community of learners;
2. collaborative approach to the work;
3. transformative intentions.

The collaborative nature of the project, its interactions with teachers and pupils and its transformative intentions permeate this thesis. The research proposal would be need to scrutinized through the University's ethical approval system (appendix 37). There were a number of ethical issues that need to be considered:

1. All the children that were to be involved were under 16 years old.
2. The use of both audio and video recording for capturing data with both children and adults.
3. The disruption to the normal classes.
4. The use of data collected from PGCE cohorts via the questionnaire
5. The use of data collected from mentors via the questionnaire
6. The use of words spoken by teachers in the semi-structured interviews.
7. Finally the privacy and confidentiality of interviews, lesson plans and informal conversations with the study school staff would need to be explained and protocols agreed.

Consent would be needed to be requested and received from the head teacher of the participating school both verbally and via email before the study can take place. However, because of the nature of the study informed consent from all of the participants (teachers and pupils) would also be needed. As (O'Leary, 2017) reminds us that consent to take part in a research study can only be given if a full understanding of the nature of their participation is explained. Verbal and written explanations of the study describing the level of participant involvement are two methods often used. In this research separate information letters and consent forms were given to the participant teachers, and the parents / carers of the pupils in the selected classes (appendix 36). These letters and consent forms outline the time commitment, the types of activities involved in the research and were endorsed by the school (head teacher) and provided to all concerned on official school headed paper. Informal consent (ie verbal rather than written) from PGCE students and their mentors was sought prior to the completion of the survey questionnaire by these two groups.

I requested and received written consent from all of the teachers who worked with me on the project. All parents / carers of the pupils involved in the two classes were contacted in writing to request their consent for their children's involvement in the research lessons, the videoing of the lessons and in the subsequent interviews. Completed consent forms were received from every parent. I considered written consent letters for pupils to fill in, but prior to the research I had spent considerable time in class talking with, and explaining to, the two classes what the research was about and their right to withdraw. As another level of safety pupils were given the opportunity to speak to their regular teacher if they wished to withdraw from the research. I did consult with the teachers and the senior leadership as to their views as to whether a signed consent form from pupils would add any additional ethical value or robustness to the research (eventually we decided against a signed form from pupils).

The pupil's right to withdraw from research was explored by Doyle (2007), citing pupil vulnerability as a major concern, with Campbell and Groundwater-Smith (2007) highlighting concerns about practitioner researchers being accountable to the pupils. So, on the morning of the lessons I explained to each class that they can ask for the filming to stop if they did not want to be filmed, additionally even though parental consent had been given they could still decide not to participate. The children were told how the video would be used, and they would be invited to watch the videos on 5 separate lunchtimes (which nearly all pupils did) and additionally how the video footage would be securely stored (and eventually deleted). As the children to be involved were only 11 and 12 years olds I was also mindful of their possible unease at expressing their discomforts about the processes, but I visited the pupils frequently prior to the classroom work so that they were familiar and comfortable with my presence in their class. Finally, I thought about what should happen if there were any behavioural incidents or unforeseen classroom problems. I decided that should there be any behavioural incidents that would require teacher invention during the filming then the camera(s) would be switched off. There were no such incidents. I was also mindful that taking an ethical approach is wider than just participant consent when Basit (2010, p. 56) reminds us that ethics "must be kept in mind throughout the study – at design stage, in gaining access to the sample, in collecting and analysing the data, in writing up the report, and in disseminating the research findings". To retain the anonymity and confidentiality of the data pertaining to the pupils and teachers involved in this study, details of which are available, these are not clearly disclosed here. Pupils, their groupings and teachers are represented using an alphabetical system.

In order to explore the influence of lesson design on the teaching and learning of fractions, I felt that classroom observation would be a key data collection tool. Classroom observation allowed me to see the reactions of teachers and their pupils, hear pupil questions and discussions and view their work. In order to carry out detailed observations and analysis I needed a method of recording this rich data. I selected video as the main data collection tool. However, such a potentially intrusive method required careful handling, not least in the gaining of consent from all parties (the university, school, parents and teachers). Sensitivity, compassion and empathy concerning the feelings of all those involved, together with the professional perceptions of the teachers were important considerations as these were fundamental if the findings were to be used effectively to inform aspects of their practice, as well as being valid and reliable for this research.

McKernan (1996) argues that the advantages of the validity, accuracy and comprehensiveness of the video data recordings out-weigh the disadvantages of editorial distortion and length of transcription. Any editorial distortion would be overcome by the employment of a number of cameras making simultaneous data recordings from differing classroom viewpoints. Using this approach allowed nearly all classroom conversations, actions and interactions to be captured. The disadvantage of the transcription of a huge quantity of data would be seen as a necessity to gain a comprehensive record; a view supported by Elliott (1991) and that the time spent transcribing the video data would be well worth the effort.

Ethically informed research should be the goal for all researchers (Blaxter, Hughes and Tight, 2002). We should be mindful that “qualitative researchers are guests in the private spaces of the world. Their manners should be good and their code of ethics and strict” (Stake, 1995, p. 447). There is, however, a greater need for ethical considerations to be managed carefully by qualitative researchers given the particularly sensitive interventions often in the personal settings of real people performing their professional duties. More specifically, researchers must address ethical concerns regarding the rights of the individuals involved (Blaxter, Hughes and Tight. 2002*)*. The ethical concerns that relate to this study concern anonymity, confidentiality, informed consent, withdrawal rights and the future use of the video footage.

3.3.8.1 Anonymity and confidentiality

Since I planned to use video as one of the central data collection methods in this study, I would not be able to maintain the visual anonymity of participants involved. I would not need names of individual pupils for this study so I did not plan to collect them; however teachers’ names were known to me but were not used. All involved were informed that lessons would be recorded as well as being analysed in order to inform the study and the teaching within the school. Written assurances to all involved were given in respect to how the video footage would be used, stored and eventually erased. I planned to share some sequences from the video footage with all members of the mathematics department as a means of prompting discussion and professional development, and as a means to gather further insights into teachers’ views and beliefs. Short video snippet highlights, were used within the school, of the teaching to illustrate effective teaching and learning and to draw attention to the distinctions between the meanings of activities, exercises, skills and tasks. The footage was used with the participating teachers on 7 separate occasions in after school discussions and at one scheduled whole department meeting. The participating school already had a policy in place regarding the use of video and photographic images for the purpose of professional development and this would be strictly adhered to during this research. Confidentiality at all times during the research would be maintained, with only single copies of transcripts, video footage, pupils' work and interview notes in the form of electronic files being stored securely. No data in the form of electronic or paper based files was shared. All materials from the study are to be destroyed five years after completion of this study. The identity of the school, teachers and pupils would not be given in this study or in any future research reports. I planned to anonymise people, places and pupil work produced when writing about them.

3.3.9 Research Timeline

The following table 3.3.9 is included to give the reader a sense of the main steps and the proposed timeline for the research study. Bowl, Cooke and Hockings (2008) remind us that the challenges around true and complete research often lead to compromise due to “different levels of teacher – researcher rapport” and the “priority given to teaching and learning issues within the institution, department or subject” (p. 90). The motivations of individual teachers and the department as a whole involved with the study may not be in concordance with those of the researcher, but in my case I had worked and supported the teachers and department over a number of years and had accumulated a degree of respect.

Time	Activity	Contributors	Comments/ Explanation
Year 1	Pilot the Questionnaire	PGCE students and colleagues.	Data not used in the PhD study – amendments made to question wording.
Years 2 to 4	Questionnaire	PGCE cohorts of students surveyed and PGCE mentors.	Data entered into SPSS and statistically analysed.
Year 3	Pilot study – based on the literature review methodology	One year 7 group with one teacher in the study school.	Ethical approval agreed. Develop learning resources and lesson plans.
Year 4	Pilot Study	The group above.	Analysis the data from the pilot study, amend the materials.
Year 5	Main research using all data tools with 2 classes and participating teachers	Pupils and teachers in the study school.	Collect teacher lesson plans written by staff prior to the research. Semi-structured interviews with participating teachers Study lesson with class 1 Follow up interviews with teachers who present Study lesson with class 2 Semi-structured interviews with participating teachers.
Years 6 - 8	Data Collection sharing: lesson videos. Thesis draft and final submission	Pupils and participating teacher.	Reviewing parts of the lessons with pupils Selected parts of videos shared with pupils and teachers to contribute to professional development of teachers. Collect teacher lesson plans written after the research.

Table 3.3.9 An outline of the main events and the time line for the research.

As previously acknowledged, and from the timeline above it is evident, that my initial starting point was from a positivist stance, with a quantitative survey questionnaire, and this almost detracts from my absolute commitment to use an interpretive paradigm as a means of understanding the relationships between teachers' lesson planning decisions and pupil learning. I did not intend to heavily rely on the results from the quantitative data as a main driver in the research design but it became apparent that the numerical data from the questionnaire did open up lines of thought that leant themselves to be explored with teachers whilst describing the social practices in the classroom. Uppermost in my mind whilst designing this research study was that "one does not begin with a theory then prove it, rather one begins with an area of study and what is relevant to that area is allowed to emerge" (Strauss and Corbin, 1990, p. 23). So the use of both quantitative and qualitative approaches would be required.

3.4 The Pilot Study

Having clarified my research questions, formulated my methodology, I realised that I would still be unprepared for the main study. I felt I needed a trial run of what I would be doing in the main study. I therefore decided to conduct a pilot study with two main objectives.

First I believed I needed to test my intended data collection methods (including the use of video) to see how pupils would react and how much data would be generated. Second, I wanted to test the teaching materials and pedagogical terminology with a small group of pupils and their teacher. The resulting video did produce a significant quantity of data which, because of a lack of clear instructions to the teacher using the video, was not always useful (e.g. poor sound quality and the videoing of irrelevancies when moving around the classroom). This generated a detailed conversation with teachers and some general guidelines for the study lesson videoing.

Conducting a pilot study as a mini-version of a full-scale study was not consider appropriate as the main concern was checking the appropriateness of the materials and resources. Piloting the data collection tools and lesson materials in preparation for the research was much more advantageous. So a pilot study would enable me to try out the research techniques and methods and still allowed me the time and the opportunities to adapt and modified accordingly. The pilot study, for this current research, might be viewed in terms of a try-out of the final techniques and methods and data tools and instruments. I therefore viewed the pilot study as both a feasibility study for the main study and as the pre-testing of the data instruments and more importantly the actions and reactions of those involved in the school.

It is worth noting here that changes were made to the original material. The manipulative tiles were redesigned, originally they were all copied on to plain white card and it was evident that considerable time was spent by pupils sorting them into families ($\frac{1}{2}$,$\frac{1}{4}$,$\frac{1}{8}$ ....) of tiles. All fraction tiles of a particular family were therefore colour coded (appendix 8), which solved the problem of wasted time. Changes to the wording on the four worksheets (appendices 3, 4, 5, 6) were made to make the phrases more recognisable to the words used by their class teachers. Few changes were made to the activity, skills and exercise worksheets. The wording on the task worksheet (appendix 6) was substantially revised.

The pilot lesson did not originally use a PowerPoint presentation, but comments from both the teacher involved with the pilot and the pupils indicated that this was the norm and to be expected. So, a PowerPoint was devised (appendix 9) and shared with both the pilot teacher and the teachers of 7AC and 7NR for comments. Interestingly I had perceived the optimal sequence of the four learning aspects in the pilot study as activity, skill, exercise and finally task. There was a discussion with the pilot lesson teacher as to whether this was the appropriate but no strong opinions were expressed, hence I did not change this for the study lessons. Finally I did use the pedagogical terminology (activity, skill, exercise and task) in a very precise manner with pupils and we did talk about whether this was problematic. The teacher expressed the opinion that this had caused no problems and this was also evident in the pilot video as pupils did not ask for any clarification from either the teacher or me.

As I have already discussed in chapter 2 my reading of the mathematics education literature, and my experiences of learning mathematics, had led me to believe that alternative approaches to teaching of fractions and the use of precise pedagogical terminology would be beneficial to pupils of all attainment and the their teachers.

3.4.1 Testing the strategy

At the pilot session the majority of the mixed attainment pupils settled to work on the use of the materials very quickly. There was a good level of interaction between the pupils using the materials and my role was limited to asking questions to prompt next steps in their learning. The materials were sufficiently engaging and little time to be spent off task by the pupils. The video of the one1 hour sessions provided some very interesting sequences which I was able to use to adapt the materials for the final research itself. The pilot video was transcribed but was only used to aid in the adaptation of the materials. It also prompted me to produce a PowerPoint and the teacher prompts I might want to use in the teaching during the research lessons. The plenary session used in the pilot was not the one eventually used in the final research. A discussion between pupils, myself and their teacher (who had observed the whole session) about the materials resulted in a number of significant changes being made and this was a direct consequence of what they had learnt. This was all contained on the video and was analysed in detail to make the changes to the materials.

The pilot session had revealed some minor issues with the materials and more importantly the understanding, by the teacher, of the pedagogical terminology that I had been using. However, she did recognise the level of engagement and learning that had taken place. She asked if she could have the materials, which I agreed to, so that she could try them with other pupils. She did this in the time between the pilot study and the main study. She adapted one resource but ran the session for all her classes and reported favourably about the outcomes. This convinced me that the materials and the related pedagogical terminology, based on what I had read and the conversations with the teacher, were robust. Given the minor changes I needed to make to the materials, together with the inclusion of the teacher developed resource, I was satisfied that these would be the ones I would use for the main research. I was confident that these teaching materials and research methods would produce the insights I was seeking.

In the interim between the pilot and the main study the teacher had reported back to her department about the work. I was subsequently invited to join a number of departmental meetings as a means of stimulating conversations and professional development. This obviously was what I had hoped for because it allowed me to pursue the next phase of my research project.

Overall, the pilot study session had provided some of the base line data that I needed. It helped me realise what changes needed to be made to the materials but more importantly it gave me a realisation of the quantity of data that was produced from just one video camera and the resulting conversation with one teacher. The main study was designed to have four or five cameras in each teaching session and more than one teacher would be present in the teaching session. Hence it was envisaged that there would be a resulting fivefold increase in the data from each of the two teaching sessions. I nevertheless decided that this was acceptable and worth the time that would be needed to be spent analysing the large quantity of data produced (the ways in which the data was analysed can be found in chapter 4).

3.5 Data collection tools

The selection of which data collection tools to use for the study is inherently related to the research design. The study design here is one of:

1. A survey of teacher beliefs on the teaching of mathematics as starting point (as an initial exploration exercise).
2. Teaching a lesson to two classes that is to be videoed – videoing the implementation of a new approach to the teaching of fractions.
3. Work with a group of teachers and trainees in their classrooms to investigate the influences of lesson design on the learning of mathematics using the lesson from the pilot.

Four main research instruments were therefore used in this study:

1. A self-designed questionnaire,
2. A specifically designed lesson
3. Semi-structured interviews with teachers,
4. Video recording (multiple sources) from the lesson delivery.

In addition to these main data collection tools the views of pupils at the end of each of the two lessons were capture on a small feedback sheet. Lesson plans written by teachers were collected which had been designed prior to the research and a small sample collected from those participating in the research after the study had concluded. The remainder of this section discusses each of these data collection instruments in detail and how they contribute to the study.

With vast quantities of data being generated from the classroom I needed to be clear about the sampling strategy I was going to employ. Bryman (2008, p. 415) reminds us that when thinking about the quantity and diversity of data we wish to sample that the approach should be done in “a strategic way, so that those sampled are relevant to the research questions”. A sample size of two classes containing pupils from both ends of the attainment range giving a diverse range of cases which were relevant to the research questions was sufficient rather than sampling all classes. Using more classes would have created a richer data set but pragmatism about the quantity of data (video footage) and time spent analysing, for what might be limited insights, was considered not to be necessary. However, a consideration of the attainment makeup of the final two classes was taken into account so as to achieve as broader spectrum of attainments as possible. A limitation of the sampling approach is that you are not able to generalise from the findings (Bryman, 2008) but this is neither the intention nor part of the study design. I eventually settled on the following sample sizes for this study:

Data Tool	Participants	Responses [Received] / (Requested)	
Questionnaires	PGCE Mentors	[21]	(25)
Questionnaires	Trainee teachers (PGCE students)	[201]	(201)
Questionnaires	Study school teachers	[6]	(6)
Questionnaires	Study school subject leaders	[3]	(3)
Semi- structured interviews	National figure	[1]	(1)
Semi- structured interviews	Study school teachers	[5]	(5)
Lesson plans	Written prior to research lesson whilst training	[12]	(15)
Lesson plans	Written prior to research lesson when qualified	[12]	(15)
Lesson plans	Written after the research lesson	[10]	(15)
Video Data	Cameras in lesson 7AC	[4]	(4)
Video Data	Cameras in lesson 7NR	[5]	(5)
Video Data	Clips from the videos.	[400]	(411)

Table 3.5 Data sample sizes and response rates

3.5.1 The Questionnaire

Initially I took the stance of a quantitative researcher investigating secondary school mathematical education pedagogy. I had conjectured that one of the ways in which mathematics teachers can affect the nature of pupil learning, and consequently their cognitive development, is by the selection of a particular teaching approach, such as a didactic. This resulted in the design of a questionnaire to test this hypothesis. I first administered the questionnaire to a study group of trainee and qualified teachers, then to their mentors within The University Initial Teacher Training (ITT) partnership schools. This was a purposive sample of 201 trainee and 25 practising teachers' views. Denzin and

Lincoln (2011, p. 202) remind us that qualitative researchers “seek out groups, settings and individuals where the processes being studied are most likely to occur”.

The 201 trainee teachers were invited to complete the questionnaire at the end of a lecture, the voluntary nature of the exercise was emphasised to all student teachers, and no pressure was applied to complete the questionnaire. At a mentor meeting where 25 teachers were in attendance 21 completed the questionnaire at the end of the meeting, the others declined due to other commitments.

My initial idea at the beginning of the research was to gain insight into the trainee and teachers’ beliefs about teaching pedagogy and practices because this approach potentially enabled me to quickly collect starting viewpoints. A questionnaire would be designed to gain quantitative data giving an insight into the typical practices of teachers in local schools, in a number of local education authorities and across different levels of teaching experience. A copy of the questionnaire (appendix 1) is included with more detail regarding its design in a later section. The intentions behind using a questionnaire would be to gain a broad indication of contemporary habits and that the results would place the more detailed classroom 'interactional' analysis into a wider context. I intended that this broad based survey should provide a macro-perspective of local trainee and experienced teachers’ views and predominant practices.
I began by applying a quantitative methodology to allow for a speedy, broad view that might be reasonably easy to numerically analyse and interpret. I initially held the view that teachers and trainee teachers would naturally approach the teaching of mathematics through the use of drills and the practising of skills than through using tasks. I conjectured that mathematics teachers often use a transmission pedagogical approach which is supported by mathematical skill development much more easily than mathematical tasks. Many of the survey questions (18 out of 26) were concerned with views on mathematical skills and tasks, the remainder concerned pedagogical approaches. Reflectively I came to realise the complexities of the interactions between terminology, mathematical subject content, pedagogical approaches and teacher knowledge. Eventually these complexities which were not easily catered for in the questionnaires moved me towards a more interpretative approach. The reflective realisation that my

initial draft questionnaire would not yield data on such issues, but the results did move my thinking forward and to a different set of questions which could not be numerically surveyed. Therefore, the purpose of the questionnaire in the study design was to gather general viewpoints and test some initial conjectures outlined earlier.

The questionnaire was designed to be as concise as possible, clear to follow and easy to understand. It used clear language and the two pedagogical approaches (skills and tasks) were defined using familiar mathematical examples. The content of the questionnaire was determined by areas of research and current practice in schools. The first section of the questionnaire sought general information, to elicit background information, such as degree qualification, percentage of time spent studying mathematics at degree level, extent of teaching experience etc. The second section consisted of 26 questions designed to elicit their thoughts relating to the teaching of mathematical skills, the use of mathematical tasks and issues relating to pedagogy. All 26 items used a 5 point Likert rating scale – almost never (assigned a score 1) occasionally (assigned a score 2), about half the time (assigned a score 3), most of the time (assigned a score 4) and almost always (assigned a score of 5). An example of my precise meanings of a ‘mathematical task’ and a ‘mathematical skill’ were given to aid the person completing the questionnaire (as defined in an earlier chapter). The 26 questions contained in the survey were categorised under 3 broad headings relating to tasks, skills and pedagogy (appendix 2). There was no opportunity for those surveyed using the questionnaire to give their own notions of skills and tasks or to add additional comments or explanations. The questionnaire was piloted by a small number of trainees, who were then not used for in the final survey. Amendments were be made in the light of the responses and comments from those completing the pilot questionnaire.

Likert item responses are normally treated as ordinal data, especially when using only five levels. Jamieson (2004, p. 1217) reminds us that ‘‘response categories have a rank order but the intervals between values cannot be presumed equal’’. Adjacent responses may not be considered by respondents to be equidistant. Therefore it is inappropriate to analyse responses using normal statistical methods so only basic statistics such as frequencies of each Likert scale point, therefore only the mode and mean could be calculated. These statistical measures would be sufficient for indicating general trends about participants' beliefs and views.

3.5.2 The Lesson Design

One of the aims of the study was to draw out the advantages and disadvantages of using an alternative pedagogy from the norm when teaching a mathematical topic (division of fractions). I therefore had a very clear notion of being able to "describe or tell the story" (Corbin and Strass, 1998, p. 25) with the findings emerging both from the collaborative research with teachers, the lesson video analysis, teacher lesson plans designed before and after the research and the semi-structured interviews. Cresswell (2007, p. 1) states that "a qualitative study is defined as an inquiry process of understanding" where the data from participants is collected at their normal place of work.

3.5.3 The Research Study Lesson

The research study lesson plan and resources (appendices 4, 5, 22, 23 and 24) were designed solely be me and constructed on my premise that there exists a hierarchy of intellectual demand inherent in the learning sequence. The lesson plan was deliberately not in three parts, the style teachers had been using. It was in four parts so as to focus on the four terms highlighted in the literature. This lesson was closely related to the type of learning which is expressed in the form of an activity, a skill practice, an exercise and a task. There was no preconceived notion as to an optimum sequence for the four learning episode even though there may be an optimum sequence for these learning opportunities. I eventually settled on a lesson sequence containing the four parts in the order activity, exercise, skills and a task because the order appeared to support the learning intentions. .

Part 1 - lesson introduction – **Activity** to recall prior learning and assess prior knowledge

Part 2 – An open-end **skills** acquisition problem where pupils are introduced to reformulating questions for example,

How many $\frac{1}{6}\mathrm{s\ are\ there\ in}\ \frac{1}{3}$? can be rewritten as $\frac{1}{3} \div \frac{1}{6}$

Part 3 - The demonstration and explanation of new learning using an alternative approach from the norm with the manipulation of fraction representations as the topic, followed by a short **exercise** practicing the newly acquired knowledge.

Part 4 – The final part would be a **task** based on a real – life functional problem with the opportunity for pupils to write and explore the context.

I intend to focus my investigations on all four parts of the lesson and evaluate the outcomes against eight distinct lesson features:-

A – Teacher Input – teaching / demonstrating / explaining
B – Pupil – Pupil Dialogue
C – Pupil Reasoning
D – Interventions (either teacher or pupils)
E – Pupils using the fraction representations
F – Pupil – Teacher Dialogue
G – Pupil demonstrating understanding
H – Connecting learning (eg division of numbers with division of fractions)

These eight categories of teaching strategies are based on the teaching advice given in the national framework for the teaching of mathematics DfEE (2001, pp. 26-27) so that they would be familiar and recognisable to both the teachers and pupils involved in the study.

The conjecture was that certain parts of the lesson would naturally lend themselves to one of the eight above features (A- H) with the inevitable consequence for lesson design. This conjecture was then analysed using the evidence from the videos and the work produced by pupils.

3.5.4 Semi-structured interviews

Both at the end of the pilot study and after each phase of the research I decided to hold semi–structured interviews with the participating teachers (Denzin and Lincoln, 2011) to gain a more in-depth view of how teachers design lessons, and the factors they consider. The following broad themes were explored in the interviews:

1. What are your views about mathematical tasks and their value for pupil learning?
2. What are your views about routine mathematical exercises or skills?
3. What are your views concerning pupils engaging in collaborative work?
4. What do you consider to be the main features of mathematical tasks?
5. What do you consider to be the effects of a teaching approach to be on addressing pupil misconceptions?

6. What were their views about the lesson structure and the methods used to teach the division of fractions?

The format of the interviews was open-ended using the themes above; with some predetermined questions to begin the conversations. The approach was semi-structured so as to encourage participants to explain their views in detail. Participating teachers are accustomed to this approach as this style is the norm in discussions after formal lesson observations (both whilst training and in performance management thereafter). Participant teachers were encouraged to be open and honest as I adopted a conversational, inclusive style rather than a formal series of answers to a straight predetermined series of questions (O'Leary, 2017). The flexibility of using semi-structured interviews (Drever, 1995), together with a sample size of five participants, was considered to be sufficient to produce useful qualitative data because Alvarez and Urla (2002) had warned us against small sample sizes of just three or four.

In a structured interview, it is usual to formulate detailed questions before the interview whereas; "semi-structured interviewing starts with broad and more general questions or topics" (Arksey and Knight, 1999, p. 5). All five participants were well known to me as I had regularly worked with these teachers over the last five years; thus a two way rapport already existed. All eleven members of the department had been invited to participate but just five wanted to be involved; availability of their time being the deciding factor. This relationship could have been considered an issue but it allowed for open, frank and honest replies and the opportunity to ask more in depth, thought-provoking, additional questions. This deeper exploration gives the interviewer the control but might restrict the interviewee freedoms in their responses (Drever, 1995) and this freedom is a key to gaining a profound understanding. The lack of structured predetermined questions also has the potential for the interview to become unfocused and hence not produce useful focussed information. Kvale (1996) reminds us that there are nine broad categories of question (appendix 39) asked in qualitative interviews to focus the interview. I decided to use these as the basis for my questions in the interviews with participants, together parts of the videos as a visual stimulus.

Parts of each video were used with both the participating teachers and the wider department to promote pedagogical discussions and provide professional development, and these were in addition to the interviews. Therefore I decided to use semi-structured interviews because I could explore themes such as pedagogical terminology, aspects of learning and teaching whilst still including some prepared questions such as those above. My justification for this approach was that being well known to the teachers I needed to be well prepared and appear to be a competent researcher rather than just engaging in an informal conversation which may not have yielded any useful data. An additional justification for the use of semi-structured interviews is to allow participants the freedom to express their views in their own terms whilst allowing flexibility for those involved to follow avenues in the conversation. A common practice is therefore to lead with open-ended questions and then spontaneously devise follow-up supplementary questions to draw out more specific evidence.

3.5.5 Lesson Video

Classrooms are complex environments with thousands of interactions taking place in every lesson which yield vast quantities of data. A single static camera placed at the front of the classroom with a view of the teacher delivering the lesson would give a restricted account of the interactions. So, to capture as rich as possible dataset I decided to have multiple cameras in each lesson. A static camera was placed at the front focused on the teacher and the presentation board. Additionally each teacher in the room (four in class 7AC and five in class 7NR, because of the availability of staff) was given a camera and asked to video just the working pairs close to where they were standing, each teacher being allocated 6 to 8 pupils.

Using video as a means of capturing data for social science research has both strengths and weaknesses. Video footage can capture fine details in social interactions which would possibly go unobserved. The accuracy of information of simultaneous verbal and nonverbal information is of paramount importance when the aim of the research is to inform and transform practice. Video allows the researcher the facility of being able to revisit real-time data a number of times and from different viewpoints. Recordings can be used to inform discussions between researchers in collaborative analysis whilst additionally providing the opportunity to notice subtle differences (Ulewicz and Beatty, 2001). Video analysis can help “to reduce the dependence of the observer on premature interpretation” and “the dependence of the observer on frequently occurring events as the best source of data” (Erickson, 1992, p. 210).

Ulewicz and Beatty (2001, p. 11) warn of an “exaggerated sense of confidence” offered by utilizing video as it is easy for researchers to believe they have fully captured the context. Whereas, Erickson (1992) suggests that video can cause a degree of embarrassment from a lack of confidentiality and the potential for a decrease in active participation. Fasse and Kolodner (2000, p. 196) identify that videos are

1. An archive for substantiating and revisiting findings.
2. Recordings are as useful micro-ethnography.
3. For use as examples of practice during later teacher professional development.

Naturally occurring data in busy, highly complex classrooms can easily be overlooked. The use of video to capture rich, thick data from the fieldwork allows the future revisiting during analysis, and the ability to describe and evaluate the context during the research cycle (Denzin and Lincoln, 2011). I had already decided to limit the amount of video footage for each of the two lessons in an attempt to both maximise the coverage and minimise the colossal amount that would need to be analysed. I initially, during the pilot allowed those videoing to record whatever they perceived as important, but after a period of reflection the main fieldwork used as suggested by Jewitt (2012, p.18)

1. a fixed video to record the whole class
2. a fixed video focused on small pairs changing every 15 minutes
3. a mobile camera focused on and moving around with the teacher

Thinking reflectively and carefully about the structure and content of the research lessons (see lesson plan appendices 4 and 5), the coding of the video data, which can often be highly theoretical I decided on a systematic, structured method of analysing the data produced. All work created from each phase of the lesson by the pupils would be collected and discussed with the teachers after it has been analysed.

It was not anticipated that pupils would receive individual feedback. It was agreed by the teachers involved that this approach would guard against the potential problem of confusing pupils who would have previously experienced a different algorithmic approach to the division of fraction which is normally taught by teachers in the department. Also feedback of perhaps a differing nature to the normal prescribed school format in school policies was not seen to be helpful to pupils. So no formal written feedback was given on the work produced but verbal feedback was given to the pupils during the lesson in the form of praise and encouragement.

3.5.6 Use of video and data authenticity

The use of videos in classrooms, irrespective of the video quality, does not guarantee useful data (Erickson, 2006). The decisions as how to best use video information to gain useful data is guided by the research aims and the eye of the observer. Although videos are a useful tool to observe the classroom environment, they also unavoidably interact with that environment. The extent to which their presence changes the behaviours of the pupils cannot be determined beforehand (Ruhleder and Jordan, 1997). Issues concerning the authenticity of the data are not going to disappear completely, but there are measures a researcher can take to alleviate some of the problems. For example the two classes in this research had had some of their lessons observed and videoed prior to the research. A method advocated by vom Lehn and Heath (2007) is to have hidden camera so that pupils would be completely unaware of the presence, however, this is not an option in pedagogical research for ethical reasons; there can be no “candid camera” classroom studies and this method was not even considered in this study.

I therefore had some concerns as to the potential effects on the quality or authenticity of the data and issues that would arise from the use of video cameras with pupils. Schuck and Kearney (2006, p. 458) had reported that the introduction of “video cameras into classrooms usually gained some attention from the students, who often behaved differently for the camera than they might have had the researchers just been observers sitting in the classroom”. From the pilot study it was identified that data obtained by zooming in on pupils was affected due to changes in behaviours, a finding echoed by Derry’s (2007) guidelines around the use of video research in education. It was therefore decided that useful data from the main study lessons would be gathered with a fixed camera, positioned at the front of the class, and a number of hand held cameras all with no zoom facilities. Ratcliff (2003) had found that pupils often act for the camera but as they get more accustomed to the presence of video cameras (Heath, Hindmarsh and Luff, 2010), they become more involved in what they are doing rather than changing their behaviour for the presence of a camera (Pink, 2007; Gobo, 2008). This was exactly the findings that the study witnessed. Additional data sources are therefore crucial in helping to contextualize what was happening on the video and “to maintain the authenticity of what was taking place in the classroom” (Fitzgerald, Hackling and Dawson, 2013, p. 54). To assist in ensuring the quality of the data from the videos it was supplemented with field notes, interviews with the teachers, and samples of work produced by pupils.

3.6 Data Analysis

This section describes how the data captured from the qualitative and quantitative measuring instruments was analysed. The transparent demonstration of the analysis process allows the reader to view how the data has been interpreted when identifying trends and relationships. The analysis of the quantitative data from the questionnaires was followed by an analysis of the qualitative data taken from the classroom work, teacher interviews, pupil feedback and lesson plans. It is important to remain aware of the fact that the data from the quantitative and qualitative sections are interconnected, in that the results of the quantitative data informed the development of the qualitative research. In that respect the quantitative data was analysed prior to the collection of the qualitative data.

3.6.1 Analysis of the Data

Marshall and Rossman (1999, p. 150) describe data analysis as “the process of bringing order, structure and meaning to the mass of collected data. It is described as messy, ambiguous, time-consuming, creative and a fascinating process”. The analysis of data invariably often does not proceed in a linear manner. It is a process of making sense, interpreting and theorizing about what the data is suggesting (Schwandt, 2007). Being from a mathematical background a logical, clinical, statistical approach to the data analysis seemed the obvious way forward, however, I soon found this to be an unhelpful course of action.

Verma and Mallick (1999) and Morrison (2012) remind us that data obtained from qualitative instruments require an interpretive approach with Baumfield, Hall and Wall (2008, p. 23) stating that “qualitative data tends to support inductive reasoning as a basis for interpreting knowledge and constructing meanings”. Very often a researcher relies on their experience of a particular setting to be able to read, understand and interpret the data. Whilst this study does use a mixed method approach to the data collection, it focuses on the acceptance of a pragmatic position and uses a phenomenological approach for the data analysis. A pragmatic phenomenology allows not merely the opportunity to observe reality through a subjective data analysis but also to adjust and change the experience of reality for others involved. This approach is justified by its usefulness. That is to say we are not looking to see if some belief or approach is ultimately true but rather looking to see if it is justified by its use. Therefore, the adoption of a pragmatic phenomenological approach to the analysis of the data allowed me the opportunity to research, observe and change practice. The final point here is the very essence of qualitative case study.

3.6.2 Analysing the questionnaire responses

The responses from 201 trainee teachers, 21 subject mentors, 3 school leaders and 9 departmental teachers from the study school were entered into a data file using the SPSS (Statistical Package for Social Scientists) programme. The contextual data from the first part of the questionnaire was also entered as free text. Each of the 26 survey questions were coded (‘S’, ‘T’ or ‘P’ and given a number 1-5 to correspond with the likert scale response). The coding became the 'case variable' in SPSS and was given a label to enable rapid logical retrieval (appendices 1, 2). Codes were also used for degree qualification subjects, gender and age for the ease of analysis. It would have been perfectly possible to have gained a huge array of very powerful statistical measures from the SPSS program, but having decided to use a pragmatic, phenomenological approach to the data analysis I decided to look for trends and the interconnections in the quantitative data. So, simple statistical descriptive measures such as frequencies, means, modes and standard deviations were all that I eventually required such as

Qu. No.	Category	Question Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	0	17	61	90	33
2	S	I think learners gain mathematical insight from practising skills.	3	53	51	71	23
3	S	I think learners should mainly work on their own when practising skills.	18	83	77	21	2
4	T	I think learners should tackle tasks.	0	14	52	85	50

Part of Table 4.9a - Frequencies Table. All trainee respondents (n = 201)

and

Qu. No.	Category	Question Text	Mean	Mode	Std. Dev
1	S	I think learners should spend time in every lesson practising mathematics skills.	3.69	4	0.8452
2	S	I think learners gain mathematical insight from practising skills.	3.29	4	1.0278
3	S	I think learners should mainly work on their own when practising skills.	2.53	2	0.8368
4	T	I think learners should tackle tasks.	3.85	4	0.8761

Part of Table 4.9l – (Mean, Mode and Standard Deviation, n=201)

The full analysis is show in tables 4.9a to 4.9v in the tables section of the thesis.

3.6.3 Analysing the videos

In total there was over 10 hours of video material from the two study lessons. The cameras were hand held by participating teachers, with each teacher being allocated a number of pupil pairs to video. In the main they spent only 2 or 3 minutes with any pupil pair before moving on. I nevertheless watched the videos from each of the cameras on a minimum of four occasions, once to transcribe the pupils' conversations and then to identify and measure other aspects of each child's interactions (with the other pupils and teachers) and learning patterns. After each transcription was written, each pair of pupils was observed in turn from the videotape. Every interaction with the materials or as conversations between each pair of pupils or pupils and teacher was noted on the observation sheet with the transcription of what had been said or done (appendix 31). I used a quantitative method to analyse the learning behaviour patterns of the pupils. The behaviours of each pair of pupils in the four parts of the lesson (activity, skill, exercise and task) were then coded using a letter system 'A' to 'H' (appendices

33 and 34). An arbitrary time of five seconds or more was arrived at from the viewing of the videos as being a realistic amount of time that a particular observation could be coded as being ‘A’ to ‘H’. The two lessons were then analysed separately using exactly the same routines and coding mechanisms. Finally each video segment included a number of pairs of pupils so the results were then combined and summarised for each of the cameras (appendices 33, 34).

3.6.4 Analysing the semi structured interviews

Having made precise transcriptions of all of the semi-structured interviews I was mindful of Schmidt’s (2004, p. 253) warning that

> The analytical techniques that are selected for semi-structured interviews within the framework of an investigation will depend on the goals, the questions and the methodological approach – and, not least, on how much time, research equipment and human resources are available.

Nevertheless, Bengtsson (2016, p. 8) reminds us that qualitative research contributes to an understanding of context, but “there is no perfectly designed study, and unexpected events will always appear”. However, irrespective of the design the processing of the data reduces the volume of text collected, identifies and groups categories together and seeks some understanding of it. In some way, the researcher attempts to “stay true to the text and to achieve trustworthiness” (Bengtsson, 2016, p. 8). A four staged approach to the analysis of textual data consisting of decontextualisation, recontextualisation, categorisation, and compilation is advised by Burnard (1991) and Berg (2001). Stage 1 (decontextualisation) involves familiarisation with the data and reading through transcribed text to “obtain the sense of the whole” (Bengtsson, 2016, p. 10). Stage 2 (recontextualisation) everything is re-read. At stage 3 (categorisation) the data is synthesised into categories to condense the data without loss of content. Stage 4 (compilation) involves the analysis and writing up process and making sense of the data in an “attempt to find the essence of the studied phenomenon” (Bengtsson , 2016, p.11)

It was this approach I eventually used for the analysis of the data from the teacher semi- structured interviews, the pupil consultations and the video transcripts. I also chose to use this approach to analyse the teacher lesson plans as these are what Wolff (2004, p. 284) would call documents or “standard artefacts” or textual material.

3.6.5 Pupil feedback

At the end of each lesson the pupils were ask to give feedback on the lesson using a prepared document (appendices 10, 27, 28 and 29). The pupil consultation was designed to inform the teachers as well as valuing the contributions from pupils to the research. Flutter and Rudduck (2004, p. 5) remind us that the premise behind involving and consulting pupils rests on “the principle that pupils can bring something worthwhile to discussions about schooling and learning”, with Lee and Johnston- Wilder (2013, p.13) noting that the “pupil voice has a vital part to play in the continuous improvement of teaching and learning in mathematics”. These comments were foremost in my mind when designing the research instrument to be used with pupils as learning involves not only the teacher but also the pupil.

3.7 Summary

The literature review, in the previous chapter, has obviously positively influenced this research in a number of ways. Firstly, it has provided a much deeper understanding of mathematical language for designing lessons. The distinctions and subtle differences in meanings had been explored in the literature and these meanings have allowed me to further refine these for the basis of this study. Additionally, reviewing the literature on the teaching and the understanding fractions revealed that the success of selecting a suitable teaching approach for this conceptually difficult topic is still debatable and controversial.

This chapter therefore outlines my ontological beliefs as a mathematician and mathematics educator embarking on a qualitative research study. The chapter also explores my epistemological stance on the learning and teaching of fractions through a collaborative qualitative case study methodology. In addition the chapter sets an ethical framework for the study.

Selecting a mixture of methods approach for the study’s data collection tools together with the analytical techniques and the use of a single school was important to keep the research well focused on the research questions. Semi-structured interviews, unstructured videoed classroom observations and documentary analysis are used to triangulate data, using my own coding systems, categorical aggregation and pattern establishment to analyse both qualitative and quantitative data. While doing so, I made attempts to enhance the validity of the study whilst adhering to and observing ethical guidelines.

I have argued that the pedagogical terms used by mathematics teachers when conversing about lessons are complex and a single school setting case study allows for conversations and understandings of such terminology to be shared and explored in order to maximise pupil learning.

It is hoped that from the review of the literature together with the selected methodological approach that a rich data set would be achieved in order to explore the 2 research questions.

Chapter 4 – The Findings (part 1)

4.1 Introduction

This chapter, and the next one, analyses the findings from all of the data sources including questionnaires, research lesson materials, lesson video, semi-structured interviews with teachers and lesson plans to address my two research questions:-

(RQ1) What are the influences of lesson design on pupil learning?

(RQ2) What are the implications of a change of approach to teaching fractions for teacher training?

This chapter will focus on research question 1, exploring the data for factors that influence participating teachers' lesson design and how an alternative lesson design impacts on pupil learning. Chapter 5 analyses the data in respect of a change of teaching approach. This chapter starts with an overview of how the data from various instruments was analysed before presenting the findings for the first research question.

4.2 The school and participant biographies

In order to locate the findings relating to lesson design in a context I now present a very brief overview of the school's mathematics department. This will then be followed by short biographies of the four participants involved in the research whose beliefs about teaching mathematics and lesson plans I discuss later.

4.2.1 The Research School's Mathematics Department.

The school's mathematics department consists of 11 teachers, 8 females and 3 males (tables 3.3.1 and 4.2.2). Only two of the teachers have mathematics related degrees and 10 teachers are younger than 35 years of age. The department has worked hard to increase the status of mathematics with parents, pupils and the wider community. In the last five years it has developed from being an inward looking department who found difficulties recruiting suitably qualified teachers to being a stable, fully staffed outward looking department engaging with the wider aspects of the profession; such as initial teacher training (Roberts and Foster, 2015).

The participating school is an active member of the University Initial Teacher Training Partnership. Given the proximity of the school to the University and the fact that the school is actively involved with initial teacher training, at all levels, it is therefore not surprising that the vast majority of the mathematics teachers are post graduate alumni of the University. Surprisingly none of the eleven teachers were undergraduates at the University. It is also interesting to note that seven of the nine teachers needed to undertake a full-time 36 week mathematics conversion course, prior to starting their teacher training as their degrees were deemed to contain insufficient mathematics.

4.2.2 Individual Participant Biographies

From the nine teachers involved in the research (table 4.2.2 – teacher F and K were not involved) five self-selecting participant teachers (teachers C, D, G, H and I) all had the same reason for wanting to be included in the classroom research. Each teacher was about to embark on the final third of their post graduate level qualification which involved a dissertation based on a classroom piece of research. All had recently studied the research methods module and had during their initial training course undertaken a small piece of classroom based research as part of the post graduate qualification. Since completing this research study all five teachers have successfully concluded their studies and gained a post graduate level qualification. Their mathematical backgrounds and teaching experience can be found in table 3.3.1.

At the time of the research all five members of staff taught classes across the years and attainment range. Eventually, by mutual consent of all involved, three teachers decided to commit to the classroom research. The classroom research used a class taught by two of these teachers. Professional biographies of these three teachers (Andy, Sarah and Tim pseudonyms) and my personal biography are included as a means of defining participants' educational backgrounds and expertise. The biographies are also examples of the range of experiences and backgrounds of the participating teachers and how their views and beliefs have changed since gaining qualified teacher status.

The three selected teachers, who are representative of the department, are used as examples to demonstrate the differences in views and beliefs held by departmental members. The three teachers had very different school mathematics experiences; this was due in part to the time when they were educated. The one teacher in the department (Tim) with a mathematics degree did not want to be fully involved in the classroom study, however he was happy to participate in a limited way such as being interviewed and providing lesson plans and a biography for analysis.

4.2.3 Biography – Andy

Andy, a mature career changer, qualified as a mathematics teacher six years ago after a very successful career in the music industry. He has a degree in theatre studies. As a direct result of his highest mathematics qualification being GCSE he was required to take a full-time mathematics course immediately prior to his teacher training course; he therefore took two years to qualify. On qualifying as a mathematics teacher his first post was a one year appointment in an academy which immediately preceded his current post in the research study school. Andy's own mathematical experiences at school were during the period when public examinations included extended coursework tasks. These tasks were used to assess one component of the then newly introduced GCSE examination. Having completed his compulsory education at the age of sixteen he left school and began work only to return to part-time study for a degree.

His views and beliefs about teaching mathematics, expressed at an early stage during the teacher training course were that, the practising of mathematical skills should be the occasional focus of lessons and that these newly acquired skills almost always led to pupils gaining mathematical insights. He did not view tasks as coursework in disguise, but he did think that tasks could be time consuming to organise and for pupils to complete.

From a piece of Andy's academic work, written during his teacher training year, his beliefs were that good mathematics teaching should be based on five key principles:-

1. Create links between mathematical topics and other subjects
2. Encourage understanding
3. Expose misconceptions

4. Use effective questioning that promotes discovery rather than teaching tricks
5. Allow and expect learners to use prior knowledge

When questioned again during the classroom research, in 2014, Andy had modified his views indicating that he now believed that the practising of mathematical skills should be the focus of all lessons even if this did not result in pupils gaining mathematical insights. He also at this stage viewed tasks as coursework in disguise which did not fit easily into the one hour lesson structure and that they were time consuming. He had completely changed his views on the usefulness of sharing lesson objectives with pupils from this being a highly positive to a highly negative strategy. Even though he had modified his views his five core principles for good mathematics teaching remained unchanged.

4.2.4 Biography – Sarah

Sarah qualified as a mathematics teacher four years ago after studying for an accountancy degree and moving from a short career in the finance industry. Whilst at school she studied mathematics to GCE ‘A’ level, but because her degree contained very little mathematics she was also required to take a full-time mathematics subject knowledge course immediately prior to the teacher training course, she therefore took two years to qualify. On qualifying as a mathematics teacher her first post was a one year full-time permanent appointment in an academy. This immediately preceded her current post in the research study school. Sarah’s own secondary school mathematical experiences excluded the long extended coursework type tasks. This was because the public examinations at the time when she took them were modular courses where she had only been required to study the subject in relatively self-contained topics.

Sarah’s views and beliefs about teaching mathematics, sampled during October of the teacher training course, were that tasks were not coursework in disguise, nor were they time consuming to organise. She thought that coursework type tasks were a very positive strategy for teachers to use when teaching mathematics. She also thought this was a direct result of the full-time subject knowledge mathematics course she had studied prior to beginning her teacher training. The predominant pedagogical and learning approach of the subject knowledge course had been through the extensive use of coursework type tasks. Sarah had therefore experienced two very different approaches to learning mathematics, one based on a formal modular content driven curriculum and the other based on an investigative open-ended task approach to teaching and learning mathematics.

From a piece of Sarah's academic work written during the teacher training course her rationale for good mathematics teaching was that the subject should be considered to be a tool for solving problems and that problem solving skills are essential for everyday decision making processes. She considered mathematics to be an essential part of our lives which involved the acquisition of basic skills and knowledge.

When questioned again during the research, in 2014, Sarah had completely modified her views indicating that she now believed that tasks were coursework in disguise which were time consuming and did not result in good pupil learning. She had therefore completely changed her views on the usefulness of tasks, but surprisingly during the research she made no reference to usefulness of practising skills which was the central tenant of her initial academic rationale.

4.2.5 Biography – Tim

Tim qualified as a mathematics teacher five years ago after gaining a first class honours degree in mathematics from a redbrick university. His own school experiences of mathematics were in a selective grammar school where he achieved 12 A* grades at GCSE and 3 'A' level all at grade 'A' which included mathematics and further mathematics. The public examinations were modular in nature which excluded the long extended coursework type tasks. Immediately prior to beginning his teacher training year he travelled the world for two years. On his return to the UK he spent six months working in a number of local secondary school mathematics departments.

Tim's views and beliefs about the teaching of mathematics, sampled during October of the teacher training course, were that tasks are time consuming to organise, whereas exercises that gradually increase in difficulty build learners mathematical confidence better than tasks. However, there were some inconsistencies in his views as he also believed that pupils should tackle tasks all the time.

Tim’s rationale for good mathematics teaching was from a purist stance where the love and beauty of the subject were the main reasons for the inclusion of the subject in the curriculum. He considered mathematics to be an essential communication tool with a unique and unambiguous language.

When questioned again during the research Tim held a firmer view that tasks take up too much teaching time and that exercises that gradually increase in difficulty were almost always better than tasks. He had changed his view about pupils tackling tasks all the time to pupils should only occasionally tackle tasks. He had therefore, after four years of teaching, resolutely remained true to his original rationale expressed during his training year. His beliefs concerning pupils experiencing tasks were also changing to be more in line with his rationale.

4.2.6 Biography – Mike (me)

I qualified as a mathematics teacher in 1973 immediately after studying for a mathematics degree. Whilst at school I studied pure mathematics, applied mathematics, further mathematics, music, physics and history at ‘A’ levels after ‘O’ levels. On qualifying as a mathematics teacher I took a one year post in a primary school before moving to a full-time appointment in a bilateral (grammar and secondary modern) high school. My own grammar school mathematical experiences were traditional in that I was prepared for examinations by being required to complete long, repetitive exercises of questions which gradually increased in difficulty. The lessons followed a format of being shown how to solve some examples by the teacher and then sitting in absolute silence to complete set exercises with no interaction between members of the class or with the teacher.

My initial views on how mathematics should be taught began to formulate during my training year where practical work, which involved dialogue and discussion were the norm and were encouraged as the expected pedagogical approach by the lecturers. During this period schools could ask their examining boards to examine pupils on a syllabus of their own design. The School Mathematics Project (SMP), started by Bryan Thwaites, at Southampton University, and The Midlands Mathematics Experiment (MME), organized by Cyril Hope, at Worcester College of Education, were just two such experimental mathematics curricula. Adapting my initial views on how mathematics should be taught is not surprising as my teacher training lecturers were the team, under the leadership of Cyril Hope, who developed and wrote the MME curriculum.

During my time spent teaching the landscape of school mathematics was changed and influenced by a number of important reports (Cockcroft, 1982; Swann, 1985); and major curriculum initiatives School Mathematics Project (1961); Curriculum Matters – Mathematics (HMSO, 1985a, 1985b); Cognitive Acceleration in Mathematics Education (Adey, 1988) and changes to the examination system with the introduction of GCSEs and the SATs (Dearing, 1994). All of these helped to shape my views and beliefs that mathematics should be taught in a completely different way to that in which I had experienced at school.

I eventually came to the belief, which I currently still adhere to, that good mathematics teaching and learning should be based on a number of key principles:-

1. Mathematics teaching needs to use a variety of pedagogical strategies and worthwhile tasks; appropriate activities and exercises are needed to develop mathematics skills.
2. Mathematics teaching has to build on pupils' thinking.
3. Mathematics teaching exemplifies and develops links between mathematical topics and other subjects
4. Mathematics teaching develops the careful and precise use of mathematical language by both the teacher and pupils
5. Mathematics teaching exposes misconceptions and enables pupils to reformulate their understanding
6. Mathematics teaching encourages mathematical understanding rather than the selective recall of facts
7. Mathematics teachers need both good subject knowledge and pedagogical knowledge.

4.3 Teachers' views that influence lesson design

It became apparent as the research progressed (both in the questionnaire data and the data from the school study) that three broad themes were emerging that might have an influence on the design of lessons. The emerging three themes were:-

1. School mathematical experiences - in two distinct groups those who were educated:

whilst coursework or extended tasks were part of the GCSE examination assessment.

prior to the introduction of coursework tasks at GCSE and were assessed by a terminal examination only.

2. Degree Type – either mathematical or non-mathematical.

3. The influence of departmental managers.

It was my initial conjecture that the first two broad themes of school mathematics experiences (tables 4.3.2d and 4.3.2e) and the degree type (tables 4.3.2b and 4.3.2c) would be important in shaping the respondents' views and beliefs and possibly have an influence on their lesson design. A third theme relating to departmental management emerged from the research undertaken in the school (table 4.3.4c). These three themes are the focus of sections 4.3.1 to 4.3.3 below.

A further theme relating to gender was investigated for differences in beliefs (table 4.3.2a) and views concerning lesson planning, however no significant differences were detected. Where gender views do differ they are highlighted as a contributing factor to above three themes. All the raw quantitative data from the survey questionnaires can be found in tables 4.3.2 to 4.3.4 and the appropriate statistical analysis measures are presented in tables 4.7 to 4.9.

4.3.1 The Influence of school mathematics experiences on the views and beliefs when designing lessons

I conjectured that there might be a difference in the views and beliefs concerning the features of lesson design based on a teacher's own school mathematics experiences. Teachers educated prior to the introduction of extended coursework tasks in the 1980s as an element of public examinations assessments were likely to have experienced a different style of mathematics lesson to those educated more recently. The views expressed in documents such as the national curriculum (DfE, 2014), the Cockcroft report (1982) and Smith report (2004) were that the teaching of mathematics should be encouraged and grounded in a "using and applying" pedagogical approach. This approach, and methods of delivering the curriculum content, relied heavily on tasks which were in complete contrast to previous methods where the repetition and practising of skills and textbook exercise questions was the norm. In 1984 the introduction of a new examination system, by the then secretary of state for education (Sir Keith Joseph), included coursework tasks which encouraged teachers to adopt a different teaching style. With the introduction of compulsory coursework task assessments and the move towards a task orientated curriculum, I therefore conjectured that the younger teachers in the survey were more likely to have experienced a different style of mathematics teaching to that of their older colleagues.

4.3.2 Beliefs surrounding designing tasks and lesson structure

Beliefs surrounding designing tasks were less marginalised between the two age groups than I had expected. The only group of teachers, from the questionnaire survey, that thought that tasks were time consuming were females educated before the introduction of a task based curriculum. This viewpoint was related to a question concerning lesson structure where all teachers educated prior to the introduction of tasks believed that the recent introduction of a lesson format consisting of 3-parts (starter, main and plenary) was a severe restriction on learning most of the time. All other recently educated teachers took completely the opposite view.

All of the mathematics teachers in the research school, irrespective of when they were educated, expressed the view that tasks should nevertheless be part of the lesson design.

4.3.3 Beliefs surrounding skills and exercises

Practising skills as a method of learning mathematics gave two opposing viewpoints in the original questionnaire. Teachers educated from pre and post curriculum changes held opposing view-points in relation to mathematical insights being gained from practising skills. Those teachers educated recently after the removal of examination coursework tasks did not hold a consensual view concerning the need for pupils to practice skills. This was in contrast to those teachers who had experienced tasks who thought skills practice should be a feature in about half of lessons.

The views and beliefs from the questionnaire data of the teachers in the research school with regard to practising skills did change over the period of time between their training and the time of the research. Staff who went to school or who were educated during the 1980s and who initially held opposing views at the start of their careers had by the time of the research moved towards the centre ground i.e. believing that skills should be part of about half of lessons. Observations and analysis of their lesson plans certainly confirmed not only that they held these beliefs but that they were also evident in their practice. In contrast those members of staff educated more recently (i.e. post task based curriculum) were of the view that skills practice was necessary only as the teacher below explained in an interview:

Me :	Do you think there is a hierarchy in skills, exercises, activities and tasks? Or do you think they are all equal?
Teacher C: (Educated at the time of coursework tasks)	Well, my view in a general mathematical sense of teaching is that I would prefer we learn the skill and then make it much more functional. We could then use the skill in a functional context. But, I am aware that the sticking point, and I keep coming back to it, is that to do it that way takes much more time than we really have.

This teacher does not see a need for the practising of skills unless they are linked to other features of the lesson design. This is precisely how recently educated staff expressed their views of their mathematics education.

4.3.4 Beliefs surrounding lesson design

Over the last 20 years or so the UK government has attempted, through various strategies, to reform the curriculum and lesson planning. Throughout this period lesson planning has been based on suggested format of consisting of three parts starter, main and plenary (John, 2006, p. 439). Teachers enact this lesson structure in mathematics

1. A starter - perhaps an oral and mental one taking 5 to 10 minutes
2. A main segment of whole-class teaching and/or paired or group work (25 to 40 minutes)
3. A final plenary (5 to 10 minutes) to round off the lesson

(Jones and Edwards, 2011, p. 73).

This standard three part lesson was seen to be important by all trainee teachers surveyed in the questionnaire (table 4.9a; question 16) and also by all the study school teachers, possibly because this lesson structure had been recently discussed in the training lectures prior to the survey. However, by the time the trainees had become substantive members of the mathematics department they had changed their views (tables 4.9h and 4.9i; question 16). Their views were now more in line with the departmental consensual view that a lesson did not have to be in three parts. Nevertheless younger teachers in the department were still, in the main, framing their lessons in this three part structure.

A total of 29 lesson plans were collected. A typical lesson plan from teacher A (appendix 23), a professionally inexperienced but mature member of the department, seems to be taking the view point that a lesson is framed in a three parts structure (starter, main, plenary) which was the style used during her school years. The lesson plan (appendix 23) is moving towards a more episodic lesson style and this lesson style (short, single learning focus, timed learning experiences) is completely different to the structure of the lessons she experienced whilst at school. The less restrictive episodic style of lesson is more receptive to the idea of lessons framed in terms activities, skills, exercises and tasks to those conforming to the three part structure.

The teacher's use of the term activity in the lesson plan (appendix 23) suggested to me that her own school mathematical experience is an influence on the selection of lesson design features. Her use of the terms tasks and skills indicates some influence of her training (where tasks and skills had been discussed) and her use of literacy elements clearly shows the influence of departmental views and practice relating to lesson structure. This teacher did describe lessons from her own schooling that had a similar structure to lesson in appendix 22; however, in a conversation with the teacher she described her typical lesson as being

> Lesson structure needs to be a starter with learning objectives and keywords, an introductory activity, a main activity, a plenary. Class teaching occurs only in the introductory activity, in the main activity – the main body of the lesson pupils 'do', i.e. practice, the learning in the lesson and this is where the pupils get to stretch themselves. Formative assessment should be used to inform planning and teaching.

This idea of a lesson almost exactly mirrors departmental policy, yet the teacher still felt able to design and deliver lessons with a structure more akin to those from her own schooling. Yet the very same teacher in an interview after the research lesson was able to express views about the restrictive nature of the three part lesson structure (appendix 12, lines 457 – 511) and critically comment on pupil learning when an alternative lesson structure was being used.

To summarise, I had conjectured that teachers' beliefs about lesson structure would be influenced by their school experience of mathematics lessons. However, my analysis of the survey questionnaire responses, lesson plans and interviews suggested that their school experiences had limited effect on their beliefs and only marginal effect on lesson planning. Whilst all trainee teachers did consider the practising of mathematics skills as an important element in the teaching of mathematics (tables 4.3.2a row 1), they also articulated the need for the curriculum to be task based. Teachers educated after the removal of coursework attach less importance to practising skills than those educated during the task based curriculum era (appendix 12 lines 282-314). Recently educated trainees, in the main, take the opposite viewpoint to their older colleagues that skills practice should only occasionally be used as a method of teaching (and learning) mathematics (from the questionnaire – table 4.3.2e rows 1 – 4 and the interviews appendix 12 lines 293-298) .

Moreover there was an absence of what might be considered as activities, skills, exercises, and tasks in the lesson plans and in the lessons I observed, suggesting that recent training had also had little effect. What did emerge from the analysis was an unexpected factor around the level of influence that whole school policies have on the design and management of lesson planning. This will be explored later. In the next section I consider the effect of a teacher's degree subject on the beliefs expressed in lesson plans.

4.3.5 The influence of a teachers' academic qualification on lesson design and teaching

It was my initial conjecture that the content of a teachers' degree (ie. whether it was a mathematics degree or not) would have an influence on the respondents' beliefs, views and lesson design. Findings from the questionnaire were that 48% of the 201 respondents had mathematics in the title of their degree, with a difference in the percentages for females and males (44% and 53% respectively). The relevance of subject academic qualifications was noted in 2004. The then government set a target for 2014 that 95% of mathematics lessons were to be delivered by mathematics specialists. The aim of this target was the belief that better subject qualifications would lead to better teaching. By 2013 only 45% of the 33300 active mathematics teachers held a relevant degree qualification

(Governors, 2015) a statistic that is almost exactly mirrored in the study survey sample. However in the school participating in the research only one male and one female (ie .2 out of 11 teachers – just 18%) had mathematics type degrees (one mathematics and one astrophysics) and this finding is replicated in over 60% of 220 partnership schools. If, as hinted at by the government, mathematically qualified mathematics teachers improve the quality of teaching and lesson design, then this is an issue that is worthy of further investigation.

4.3.6 The Influence of non-mathematics degrees on lesson plans

Comparing the lesson plans from five teachers in the research school who did not have a mathematics degree with those written by a single teacher with a mathematics degree does not allow for generalisations to be made. However, in the main, the five teachers without mathematics degrees produced far more mathematically detailed lesson plans. Also these teachers felt the need to do worked solutions for the mathematics, as can be seen in figure 4.3.6 below.

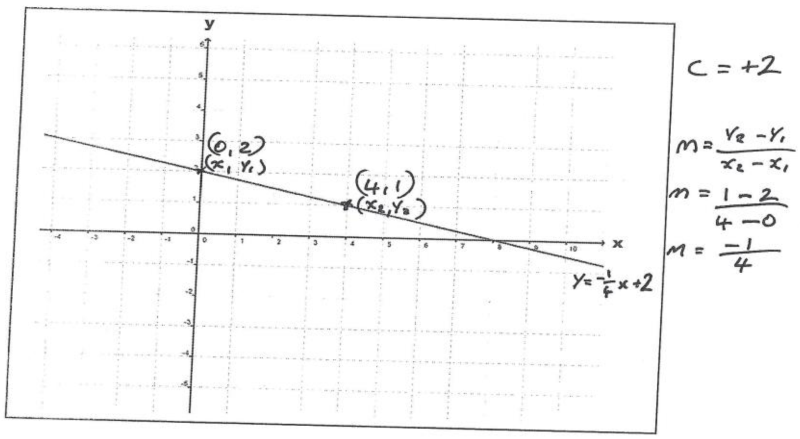


Figure 4.3.6 A non mathematics graduate notes in the lesson plan for linear graphs.

Additionally, teachers without mathematics degrees spent considerable time producing worked solutions to many of the routine questions that they were setting pupils, Lesson plans contained many worked solutions with one teacher producing solutions to every question they set. This has the obvious implications for workload.

From classroom observations of the five teachers this apparent lack of mathematical self-confidence did not result in a lack of confidence in answering questions from pupils during the lessons. In fact the time spent by the teachers producing routine solutions appeared to be worthwhile in that their resulting explanations to pupil questions were well considered and well formed. Taken from the lesson observation notes for the question in the diagram above,

Jo:	Sir, I know why c = 2 because it cuts at 2, but why is it $\frac{-1}{4}$ when it cuts at +8?
Teacher A:	That is a really good question, Jo. The line slopes downwards so what does that tell us?
Jo:	That it is minus.
Teacher A:	Well done Jo. Now look at the line for every one unit it goes down the y-axes how far does the line travel along the x – axes? (pointing to the graph on the screen)
Jo:	Four.
Teacher A:	Well done. (Writes on the board) The line slopes down means minus and for every 1 it goes down 4 means ($\frac{-1}{4}$)

4.3.7 The Influence of mathematics degrees on lesson plans

The teacher with a mathematics degree produced no evidence of any worked solutions on any of his lessons plans. This teacher, having done no worked solutions in his lesson plans, did give both mathematical precise solutions and explanations to questions from pupils. However, in a number of observed instances he failed to grasp the difficulties pupils were experiencing with his solutions and explanations.

For example here are two short interactions from different lessons:

Pupil A:	Sir a + a + a = 3a , why isn't a + b = ab?
Teacher B:	because you can't add "a" and "b", they are different.
Pupil B:	I know 10% = $\frac{1}{10}$, is 20% = $\frac{1}{20}$
Teacher B:	No it's $\frac{1}{5}$ (no explanation given)

In the main this was because a number of important steps in the explanations (solutions) were missing and in conversations with teacher B it became evident that he did not consider the steps to be necessary and implied that the pupils did not need them either.

Me:	I'm not sure the pupils will make the connection between 20% and $\frac{1}{5}$, so what were you expecting them to do.
Teacher B:	They know 20 x 5 = 100 and that percentages are out of a 100.
Me:	So, with hindsight do you think it would it have been useful to have included this step?
Teacher B:	I see little point explaining all the steps. One of the aims of mathematics is to create pupils who can think for themselves. They already have the knowledge they need from a good understanding of multiplication tables they need to apply that knowledge by themselves rather than me detailing every step.

These limited findings seem to suggest that those teachers who do detailed worked solutions tend to give more considered responses to pupil's questions (they are learner focused rather than subject focused), and that they might have a more detailed understanding of the conceptual difficulties pupils encounter.

4.3.8 The Influence of a degree qualification on practising mathematics skills in class.

An interesting, and totally unexpected response, was given to the question relating to practising skills ("I think learners should spend time in every lesson practising mathematics skills") and this was completely the opposite of what I had envisaged. In the questionnaire survey the vast majority 63 out of 96 (68%) of those with mathematics degrees did not see a place for pupils practising mathematical skills in the classroom (table 4.9f; question 1). It is possible that mathematicians have a wider view of the subject and do not see the subject in terms of facility and dexterity of manipulation of algorithms. However, when I observed them in classrooms during their training and in the classroom in the research school, I found them doing exactly what they said they did not do, i.e. getting pupils to practise skills.
Those trainee teachers with non-mathematics degrees took completely the opposite viewpoint with 57 out of 105 (54%) stating that they would have pupils practising mathematical skills most of the time (table 4.9g; question 1). When I

observed this group of teachers in their classrooms during their training and in the research school, I found the majority of their lessons to be investigative tasks with some skill practice. Again the opposite of what their original views and beliefs had been at the start of their training year.

I wanted to understand why there was a difference in lesson plans, pedagogy and views about skills practice among those with and without mathematics degrees, and to find out from the teachers what they felt about the need for a mathematics degree as a qualification for teaching the subject. The following represents their respective views:

Me:	Do you think a degree in mathematics is absolutely necessary to teach the subject?
Teacher D: (Psychology Degree)	Before I started the PGCE course my answer would have been a definite yes. However having now taught for 4 years I would definitely say no. I feel I have enough mathematical knowledge for the groups I teach and when I am meeting a topic for the first time then I have colleagues to help me. When preparing for lessons I spend a lot of time answering the questions myself – just practising to make sure I can answer any question asked of me. I feel the practising of the skill is important for me and for the pupils.
Teacher C: (Media Degree)	I did feel mathematically inadequate when I first started the PGCE course as there were people on the course with degrees in mathematics who obviously found mathematics easy. I soon realised that subject knowledge, whilst it is an important part of a teacher, it does not make you a teacher. I did, and still do, need to keep practising mathematics but this now is almost always related to GCSE topics these days. I feel less constrained when dealing with Key Stage 3 topics. Teaching is relationships with children as well as having a relationship with the subject content.
Teacher A: (Mathematics Degree)	It really depends at what level or age the lesson is being given. I think it is an absolute requirement for teaching ‘A’ level and the top set year 11. My maths degree is the only reason I am teaching year sixth-form mathematics. The mathematical skills required to answer problems at ‘A’ level are, in my opinion, completely different to those required for the years below and if you have not studied maths at degree level then you do not have the required knowledge. I do not feel the need to answer every question when looking at exercises for pupils. I tend to do the last one

> or two of the exercise. I think my subject knowledge is an important part of me as a teacher especially as I have been tasked with starting a sixth-form ‘A’ level class next year.

Subject confidence was explored in a number of formal conversations relating to the teaching of fractions and in particular the deeper conceptual knowledge of why some of the ‘tricks’ or ‘shortcuts’ in mathematics exist and can be justified (such as the ‘flip – method’ for division of fractions). Often non-mathematicians (as was the case by those in the study school) give the response that “it just is and that is the way I was taught”. They are normally unable to give a mathematical reason or an alternative method. I had expected that the mathematically qualified teacher would give a more reasoned mathematical justification of why the ‘flip’ method works, but this was not the case. As part of my role in initial teacher training I have interviewed hundreds of potential teachers with mathematics degrees and I have failed to find a single applicant who can convincingly explain or give an alternative method as to why such mathematical tricks or shortcuts work. They nearly always revert to the phrase “it was the way I was taught”.

The three teachers above obviously have differing conceptions of what makes them a mathematics teacher with the inevitable impact on their lesson planning and delivery. The first two teachers perceive themselves as teachers of children and know their mathematical limitations and how and where to gain the required subject knowledge help that they need. The final teacher would appear to view his role more through an absolute confidence in his subject knowledge. It might be concluded that the first two teachers perceive their professional identity as either teachers of mathematics or as a mathematics teacher, whereas the other sees himself as a mathematician. Bromme (1991) argues that teachers derive their professional identity from ways in which they see themselves as either subject matter experts or pedagogical experts or combinations of the two identities. Whereas Beijaard, Verloop and Vermunt (2000) redefine pedagogical experts into a didactical and pedagogical expert, thus adding a third identity:

> **a subject matter expert** is a teacher who bases his/her profession on subject matter knowledge and skills;

> **a didactical expert** is a teacher who bases his/her profession on knowledge and skills regarding the planning, execution, and evaluation of teaching and learning processes;
> **a pedagogical expert** is a teacher who bases his/her profession on knowledge and skills to support students' social, emotional, and moral development (p. 754).

Findings from this study suggest that the participating teachers do not differentiate between didactical expert and pedagogical experts preferring to view their professional identity in accordance with Broome's (1991) two broader definitions.

4.3.9 The influence of mathematics Subject Leaders on Newly Qualified and Trainee Teachers

The traditional role of a head of department has long been associated with leadership, management and administration of the department. The view of Metcalfe and Russell (1997) of the head of department being "responsible for the syllabus and ordering the stock" is now being superseded by a wider more all-encompassing definition as noted by Bennett et al. (2003) and Hilton (2017).

> It would appear, then, that subject leaders require a combination of teaching expertise, subject knowledge and good interpersonal skills if they are to obtain and maintain the authority they require to do their job (p. 7).

This view was echoed by one of the subject leaders in the research school who commented in an interview that

Teacher J: One of the aspects of my role of subject leader of mathematics includes telling younger members of staff how things should be done. I don't remember being told anything when I started teaching. More and more detail is expected from me as to how topics are taught, expected pupil misconceptions and the ways of rectifying pupil mathematical errors. When I started teaching I think it was assumed that I already had the knowledge of both the mathematics and how to teach the subject. This is a huge additional workload, and not specific to this school as colleagues in other local schools say exactly the same when we meet.

The expectation of the subject leader that is being articulated here is one of a mediator of beliefs and values as well as the more traditional subject expert.

It might be reasonable to assume, as McLeod (1992, p. 579) argues, that "beliefs are largely cognitive in nature, and are developed over a relatively long period of time". I therefore assumed that the six newly qualified teachers' views and beliefs would be broadly similar and would remain reasonably consistent with their initial views expressed in the November survey of their training year, inferring a limited impact from their departmental leaders. However, there were some significant shifts in their beliefs over the period between their training and the time of the study (average of two years). The most significant change was in their views relating to mathematical pedagogy which had become closer to the views of their subject leaders. This is not surprising because frequently, if not daily, the professional discussions between colleagues seemed to be mediating and modifying the views of newly qualified teachers as witnessed in the interviews (appendix 13 – lines 161-175) and the survey data (tables 4.9h-4.9k – question 16).

The findings relating to professional roles and responsibilities would tend to suggest that once teachers' beliefs have been established they are difficult to reformulate (appendix 13 – lines 177 -184). This can also be seen where subject leaders hold completely opposing views (for example where the modal values in the data from the questionnaires differs by at least 2) as was the case in four questions (tables 4.3.4b and 4.3.4c). These four data items, highlighted in table 4.3.4b, were of particular interest as they directly relate to lesson design, the practising of skills and the use of mathematical tasks.

The professional maturity and expertise of the subject leaders possibly explains their collective views. However, the difference in views between subject leaders and newly qualified teachers relating to the design of tasks and the differentiating of tasks is not easily explained given that they were moving towards departmental consensual views in other areas.

Two subject leaders commented

Me : When you use tasks with a class, how easy are good mathematical tasks to design?

Leader 1 : Teacher J — Well, I seldom use my own tasks, I rely on published materials mainly because of time but also I know that published materials are tried and tested and meet the mathematical content for GCSE.

Leader 2 : Teacher E — As trainee teachers often teach a reduced timetable they are able to spend time designing and trialling their own tasks and materials, as this is a requirement of their university course. Obviously the trick is to store and share all of the tasks that are generated. We have been really fortunate at this school as all of us in the department share materials, but the differentiation of a task is still left to the individual teacher and as a subject leader I rely on differentiation by outcome. So there is still a time implication when planning.

Here the views of the subject leaders around the time implications required for planning lessons is an example of the pressures that influence newly qualified teachers. Where subject leaders hold differing views to the newly qualified teachers, then the newly qualified teachers changed their views with the tendency to move towards the more polarised view of the subject leaders that they most closely associated themselves with. These changes of viewpoint were mainly evident in the questions relating to pedagogy and were partially a direct consequence of the procedures and protocols of the department. For example, the departmental lesson planning proforma (appendix 16), where each lesson was required to be centred on a teachers' predetermined view of a three way differentiated learning objective (labelled bronze, silver and gold), could be interpreted as stifling both pupils and the teachers. The influence of departmental subject leaders cannot therefore be under-estimated, however at least two of the newly qualified teachers had modified the lesson plan to add a fourth learning objective labelled platinum, so a measure of individuality from the newly qualified teachers was still able to be observed.

During this research subject knowledge or subject expertise was never discussed in either formal or informal departmental meetings. Subject leaders expressed the views that their colleagues' knowledge of the subject was sound or better and saw no reason why time should be spent talking about subject content knowledge at the expense of other administrative items. This was a surprising comment given only two of the eleven members of the department had a mathematics degree. None of the subject leaders had a mathematics degree and their assumption and beliefs were that classroom experience results in good subject content knowledge (appendix 12 – lines 316 onwards and appendix 11).

4.4 Lesson Design

I wanted to analyse the extent to which teachers' beliefs about activities, skills, exercises and tasks were reflected in their lesson plans, and if they were present in the lesson plans how they had changed after my intervention. A total of twenty-nine lesson plans from the five members of the study schools' mathematics department were analysed. Twelve plans were scrutinised from their training year, twelve plans were taken from school performance management lessons delivered around the time of my intervention and a further five lesson plans were analysed from the five teachers participating in the classroom research (appendices 21, 22, 23 and 24 for examples of these lesson plans). These five lesson plans were written, and the lessons delivered, in the three weeks immediately after the research lessons had taken place.

4.4.1 How did teachers design their lesson prior to the research?

Whilst talking with the teachers involved in the study they frequently referred to textbook exercises and mathematical skills when conversing about a mathematical operation, algorithm or the approach to the teaching a particular topic (appendices 21 and 22). So I was surprised that ten out of twenty four lesson plans written prior to this study contained no references at all to any of the four words or any synonyms or approximations to the words activities, skills, exercises or tasks (table 4.5.1) and that only fourteen lessons plans contained some references to the words skills and exercises. However, closer analysis revealed that these words were used as descriptors for “educational experiences” (such as an exercise for pupils to self-assess knowledge of ratio) rather than to define any “learning intentions” (such as pupils will develop the skill of accurately measuring by measuring items in the classroom). So the use of the words in the lesson plans had not been consistent with their use in departmental conversations, and written lesson plans were not reflecting their verbalisation of their own conceptions of the two words. This finding was totally unexpected as in observed departmental conversations these two words, skills and exercises, were consistently used and therefore I would have expected the lesson plans to reflect these conversations.

I then looked at the fourteen lesson plans for how the individual teachers used the words. Table 4.5.1 contains the details of the exact phrases and the use of the words “activity” and “task”. I then compared this with their beliefs about these terms as revealed in the initial survey questionnaire. What I found was a change in both their use and interpretation of the terms. For example, teachers A and B initially stated in the questionnaire that tasks “should be tackled by pupil almost all of the time”. However, their lesson plans suggest they had changed position, instead seeing tasks as “difficult to organise, time consuming and to be used infrequently”. During informal conversations with these two teachers it transpired that the effective use of their time, and the need to write the absolute minimum in lesson plans, was much more important than implementing their views and beliefs. Teacher B commented that

> the workload relating to administration, which has to be done and is a priority, means that a small amount of my time is spent writing lesson plans. The lesson plan often then becomes a to-do list rather than how I want to teach the topic.

For teachers A and B the lesson plan was considered to be unnecessary as they felt they had no need to refer to it during the delivery of the lesson. The lesson plan was considered to be something that they had to do and was of little value, a surprising view given they were relatively recently qualified.

In contrast teacher D did use tasks according to her initial surveyed view and the words activity and tasks were found in every lesson plan she produced. Teacher D, newly qualified, explained that she needed a detailed lesson plan as a reference point during the lesson in order to deliver an effective lesson. Additionally the two words (activity and task) were indicators for types of actions to be undertaken by the teacher as in appendix 44. This is in complete contrast to the way in which I was using the terms during this research.

Contrastingly I would have expected teacher G to have indicated in the lesson plan the unforeseen mathematics that was being sought given her surveyed views about tasks leading pupils to incorrect mathematical conclusions. The task relating to probability can give pupils lots of mathematical surprises with a number of unusual mathematical outcomes. None of these were noted in the lesson plan just in case they were not found by pupils and were needed to be introduced by the teacher.

Interestingly two of the nine teachers did not use any of the four words in any of their lesson plans. When interviewed they expressed the view that their lesson plans were more of an itemised list of jobs to be covered during the lesson (eg date, title, put objectives on the board, do example 1, textbook page 9 questions 1-10). The remaining seven teachers used the words (activity and task) on thirteen occasions to describe an action taken by pupils or by themselves.

As a means of illustration these actions the following are direct quotations from lesson plans

> Teacher A: “Teacher sets a task of differentiated questions”
> Teacher D: “Draw lines of symmetry, check pupils are on task”
> Teacher F : “Explain task …”

In only two instances did teachers use one of the two words to describe a learning intention, for example:

> Teacher B : “Pupil Task – using multiples of 10% and 5% in a shopping context to find the discounts and discuss which you would buy”.
>
> Teacher G : “Pupil Task - By the end of the next 15 minutes I would like to know the chances / probabilities of picking each colour”

Teacher D (newly qualified) appears to have differing meanings for the word task, contained in two of the lesson plans was:

> LP1**: “**Use clues to deduce age of man; give 5 minutes for the task”
>
> LP2**: “**Drawing lines of symmetry – check pupils are on task”

In lesson plan 1 the teacher uses the word task related to a period of time for pupils to complete a piece of mathematics, whereas in lesson plan 2 the word “task” relates to an action that the teacher is planning to carry out. This lack of a common use of pedagogical terminology might be a reason why teachers did not feel comfortable using them in the observed departmental conversations. This lack of a common use of pedagogical terminology restricted their observed professional conversations to just exercises and skills for which they did seem to have a degree of common understanding.

Furthermore all five teachers have slightly different meanings for the word "task". Teachers B, D and G equate the word "task" with time, teachers A and F with additional pupil work, finally teachers D and G with a learning action related to the mathematics. These multiple conceptions of the word influence their use in the written plans when designing learning sequences which is shared with pupils. This divergence in the teacher conceptions of the terminology may unwittingly have an impact on pupils when they encounter different interpretations of the words used by various teachers over their school career.

There would appear to be little evidence in the lesson plans, or the questionnaires, that teachers think about the distinct meanings of the four words, and which in the main lesson plans are used for actions rather than learning intentions. The fact that teachers interchange the meanings and use of the four words, and also totally ignore them, might indicate that they view activities, exercise, skills and tasks as one and the same.

4.4.2 How did teachers design their lessons after the research?

Having analysed the lesson plans written before my research, I then looked at five additional lesson plans written by the same teachers after the research had taken place. I found the words "activity", "exercise" and "task" in all the plans but only one lesson plan used the word "skill". Each of the words were used in an almost identical way to that of the research lesson, with the implication that teachers were beginning to consider pedagogy terminology and modify their understandings of the similarities and differences in meanings between the words. Where the two words task and activity were used they were nearly exclusively in lesson plans designed for the age range 11-14, with only 2 occurrences in lessons plans for the 15-16 year olds (public examination groups). I conjectured that this might be because this teacher believed that there is time for tasks and activities in the younger years but not so for those studying for formal public examinations at the age of 16.

Findings from the survey questionnaire did not fully support this localised school finding, however from conversations with all eleven teachers, and directly from the five teachers involved in the classroom research, they all were of the opinion that tasks are time consuming and that they often lead learners to incorrect mathematical conclusions. Tasks which resulted in pupils' making incorrect mathematical conclusions take time to explore and mediate. Their overwhelming view was that the crowded curriculum, the lack of time, and their unwillingness to engage pupils in tasks whilst studying for public examinations, was the main reasons for preventing them from designing lessons involving tasks for older pupils.

All of the five lesson plans, designed and delivered after the research intervention, were for classes of 11-12 year olds (key stage 3) and none of the plans related to the research study topic - fractions. Again analysing the lesson plans for the four words (activities, skills ,exercises and tasks) I found that they all used the word activity and task in a similar, if not identical way, to that of the research study (table 4.5.2). I was not surprised by this because part of the research design was to inform the teachers of the meanings of these terms so that they could be applied in their classrooms. The word "activity" was used to either recall learning in plans by teachers C and G, or to produce items as in plans from teachers D and G (table 4.5.2). Teachers were also more aware of, and beginning to consider how, the learning was to take place (individually, in pairs or in groups).

Strikingly, there is an obvious change in the use of the two words (activity and task), and the quality of activity and pedagogical thought related to these learning episodes planned for pupils. The word "task" was used by three of the four teachers (D, G, and H) in a similar ways to those I had written for the research lessons (appendices 7and18) where the learning episode was in the form of a real-life problem. Teacher C did change the style of the task and did not place the mathematics in a real-life problem, instead grounding the abstract nature of the topic (solving linear equations) in the context of a hypothetical discussion between two pupils. Teachers were implicitly more aware of the learning theory they were employing and were beginning to consider how the learning was to take place (individually, in pairs or in groups). The word "activity", as defined earlier and used in the research lessons, was "an open – ended learning sequence to recall prior learning". Interestingly all four teachers used this definition of the term activity in their teaching and were consistent with the research.

Again in none of the four plans were there any references to the words "exercises" or "skills", this was totally unpredicted as I was expecting the lessons to follow a similar style and pattern to my research study lesson. In informal conversations with all the teachers they highlighted the pressure of the expectations to follow a whole school approach to lesson planning (the 3 part lesson) which left insufficient time to do all aspects of the research study lesson. Additionally all the participating teachers independently and collectively voiced the opinion (during conversations and in the semi-structured interviews – appendix 12 – lines 210 to 221) that it was neither educationally desirable nor practically feasible to have all four elements (activities, skills, exercises and tasks) in every lesson. However they did recognise the value of the clear definitions of the pedagogical terms (appendix 13, lines 185-198). On reflection I considered this to be a sensible viewpoint, the lesson was never designed to be a "model" or "standard" lesson; it was designed to assess the features of the four terms and the effects of the learning sequence. No opinions were expressed by the teachers as to the order of the four learning episodes but they all agreed that either the task needed to be the first or last episode of the lesson. When questioned as to their reasoning the consensus was that a task was either appropriate to set the context for the lesson content, or to finish a lesson with a real-life application of the mathematics that had been presented during the lesson.

4.5 Summary of the Findings

Returning to the research question

> **(RQ1) What are the influences of lesson design on pupil learning?**

Research question 1 relates to the learning of mathematics and lesson design. I have therefore grouped the findings from this part of the study into the following two broad categories:

5. Learning mathematics **(RQ1)**
6. Lesson design terminology **(RQ1)**

4.5.1 Learning mathematics

Finding 1: Pupil learning (as observed in the video evidence according to the extent to which pupils reason, demonstrate understanding and make connections in mathematics) is enhanced when lessons are designed on active learning principles using the sequence of the lesson episodes activity, skill, exercise and a task. This approach also motivates and encourages pupils across the attainment range to tackle quite sophisticated real–life tasks. When pupils are able to collaboratively construct an understanding of the mathematics they are able generate and answer their own questions, this gives pupil's a deeper conceptual understanding (appendices 47 and 48). Lessons which are situated (Lave and Wenger, 1991) in real-life tasks generate a community of practice (Lave, 1988) where learners are free to pose questions, discuss their solutions, which results in a reforming of their conceptual understanding of the mathematics.

Finding 2: When using manipulatives (physical objects such as fraction tiles) pupils in the study lessons showed an increased number of attempts at posing questions both written and orally. Reflecting on the validity of mathematical solutions they also aided pupils of all attainment levels in developing confidence when learning about fractions. The quantity of interactions with the manipulatives (72 five or more seconds observed in the video data), is evidence of their influence on pupil motivation and engagement with the mathematics. The post lesson discussion with pupils indicated that the manipulatives were influential in helping to build their confidence and contributed to their success when tackling all aspects of the lesson. At one level manipulatives support the development of conceptual understanding of the relationships between mathematical operations and fractions thus aiding problem solving when tackling real – life tasks. At another level manipulatives influence pupil motivation, engagement and confidence. The impact of the use of manipulatives on learning can be seen, across the attainment range, by the sheer quantity of correct solutions and from experience this is a significant improvement when compared to outcomes from more standard mathematics teaching approaches.

4.5.2 Lesson Design Terminology

Finding 3: There are differences between the terms activity, skill, exercise and a task but these are not always clear and discernible to trainee teachers or newly qualified teachers. Teachers often seamlessly interchange the terms activity, skill, exercise and a task in their lesson plans and conversations with colleagues. The use of these terms is often to describe an "educational experience" (eg. an exercise to self-assess) rather than an "educational learning intention" (eg an activity to develop accuracy in measuring lengths). Where lesson plans are shared between departmental team members, as was the case in the study, this can result in different enactments due to teacher interpretations of the terminology used on the lesson plans.

There was a change in use of the four terms activity, skill, exercise and task in teaching created by teachers after the classroom research, their use by the teachers was more in line with that of the research lessons. However, not all four terms were used when teachers were creating lessons or teaching across the age range after the initial research.

Mathematical exercises are almost always synonymous with mathematical skills in the minds of teachers. The findings tend to suggest there is a subtle distinction between an exercise and a skill, where an exercise is the application of a skill, in this case the division operation, which results in a deeper conceptualisation of the mathematics.

A mathematical activity, where no instructions are given, allows the teacher to diagnose prior pupil learning and any misconceptions. This type of activity seems to engage and motivate pupils across the attainment range and was a direct consequence of the design which was founded on an active learning experience.

Finding 4 : The standard practice of a three part lesson (start, main and plenary) is not always adhered to by teachers. Some teachers prefer an episodic lesson design with short focussed learning episodes consisting of skills, exercises and tasks (appendices 21 and 22). John (2006) argues for flexibility in lesson planning as a requirement for teachers to develop alternative plans. With each teacher being very different, once the basic aspects of lesson planning are understood then the dominant model is probably no longer a necessity and alternatives will allow for creative and experimentation in the design of learning plans. The pre and post lesson plans designed by teachers in this research (appendices 21, 22, 23, 24 and 25) would tend to support this viewpoint. Where mathematics lessons are designed around a lesson structure consisting of an exercise, activity, skill and a task it would appear to be a structure that will support pupil learning. Where a lesson culminates with a real-life task (for example the division of fractions) this appears to support the development of mathematical understanding for pupils of all attainment levels.

The next chapter will consider the findings for research question 2.

Chapter 5 – The Findings (part 2)

5.1 Introduction

This chapter analyses the findings from all of the data sources in respect of

> **(RQ2) What are the implications of a change of approach to teaching fractions for teacher training?**

5.2 Analysis from the study lessons

This section presents the findings from the pupils' work produced in the two research lessons, video evidence, interviews with teachers and pupil feedback forms. The section is divided into five subsections detailing the findings from each of the four learning episodes (activity, skill, exercise and task). The section starts with a brief introduction and explanation of the research lesson.

5.2.1 The Research Lesson

Having researched the literature and come to conclusion about definitions for the pedagogical terminology (appendix 35) I then designed a lesson (appendices 7 and 9) to investigate the influence these terms might have on pupil learning. The lesson was in four parts: an activity, skill, exercise and a real-life task. To aid conceptual understanding each pupil pair was provided with a set of manipulative tiles (figure 5.2.1 below) representing the fractions (1, $\frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$). The tiles were available to support the conceptual understanding and learning of the division of fractions in three of the four learning episodes (exercise, skill and task). They were not available during the activity learning episode due to the design features where pupils were required to recall prior knowledge of fractions.

Figure 5.2.1: The complete set of manipulative tiles supplied to each pupil pair.

A general finding, from both classes and across the three learning episodes, was that the manipulatives tiles were considered to be supportive of r pupil learning as evidenced by the comments taken from participant teacher interviews such as:

Me:	Do you think the tiles were useful?
Teachers D and H	Yes (together). When they weren't using the tiles they seemed to get things the wrong way round; for example to get from a $\frac{1}{3}$ to a $\frac{1}{6}$ you times by 2.

and from pupil feedback forms (appendix 28) such as:

Pupil h	They showed me how many different fractions go into a whole which helped when doing the questions.
Pupil d	They helped because it had a visual and I could see what when in the GCSE question [Worksheet 3].
Pupil b	They helped me because it had a visual effect and you could see how many made a whole.

These very same findings had been noted by other researchers (Grant and Searl, 1997; Kamina and Iyer, 2009; Loong, 2014).

5.2.2 Student learning from the lesson – (Activity)

The activity learning episode (appendix 3) of the research lesson was designed as a lesson starter of approximately ten minutes in length. Pupils were actively encouraged to work in pairs (this was evident in the video evidence and the quantity of work produced in a short space of time) to work, discuss and recall their prior knowledge of fractions. The activity was not designed to introduce new learning. In this learning episode pupils were set an open-ended activity of writing down all that they knew or could recall about fractions, no hints or help were given by me or the participating teachers. Here there is no intention to make comparisons between the two classes as they were of very differing attainments, so the outcomes were expected to be different. Also this was the first time the classes had revisited the topic in nearly a year. The activity episode does however allow me the opportunity to take a view across the attainment range about prior learning and what knowledge was able to be recalled.

The pupil outcomes clearly indicate from the video evidence (appendix 32), and a sample of their written work (figure 5.2.2 – different styles of writing; top right 2 boxes are different to the others) that both pupil partners in the pair made contributions. This type of learning episode was not new to pupils in either class. It had been previously exclusively used at the end of lessons to recap or reinforce learning or as an assessment of progress during the lesson. I decided to use this style of activity as the starter to capture, motivate and engage the pupils at the outset of the lesson (Kyriacou, 1992).

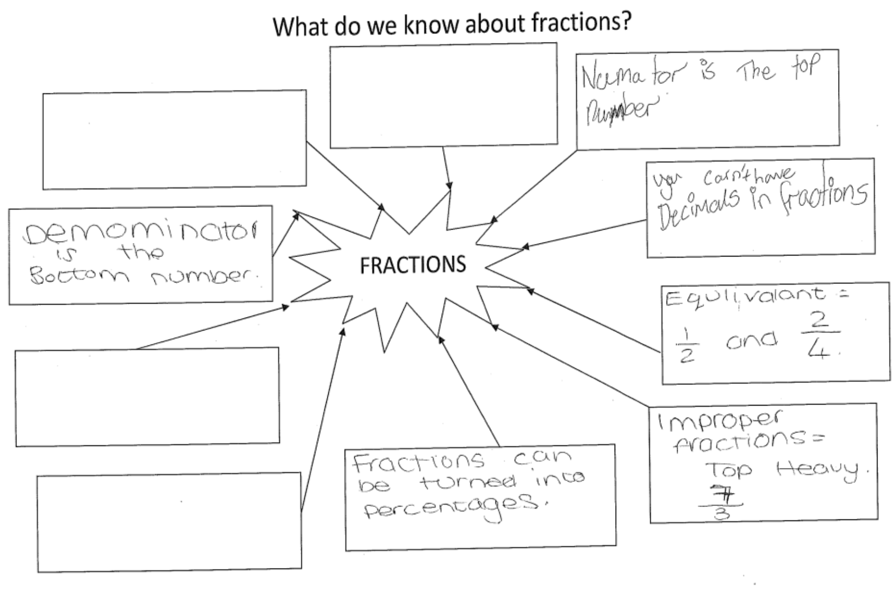


Figure 5.2.2 Example of paired activity work from a pupil pair in 7NR.

Additionally the activity allowed me to gauge and assess their levels of knowledge and diagnose any misconceptions of the subject matter so that the remainder of the lesson could be pitched appropriately. There were no major misconceptions but the activity did uncover some minor misunderstandings such as, all fractions are less than one, which were addressed in the remaining parts of the lesson. Table 5.2.1 shows how the outcomes were analysed and categorised for 7NR as well as the overall individual and overall percentages for the two classes.

The participating teachers and I had anticipated that the pupils would mainly recall fraction facts and have a large number of misconceptions. We also expected some pupil pairs would write about being able to perform operations on fractions. Additionally we expected to see some pupil pairs writing single unconnected phrases such as ‘improper fractions’ or ‘equivalent fractions’. Finally we anticipated that the number of misconceptions would be significantly greater in the lower attaining group.

In actual fact when the written work from the two classes was analysed the majority of pupils were writing about types of fractions such as improper or equivalent fractions. Some of the phrases had detailed explanations as to their mathematical meaning. The pupil outcomes from the activity either related to subject (topic) vocabulary or knowledge of the topic (i.e. mathematical terms or operations relating to fractions). Pupils were also recalling fraction facts, for example – “you can’t have decimals in a fraction”, “a half is 0.5 or 50%”. They were also describing types of operations on fractions, for example, “some fractions can be cancelled down”, “you can add and take away fractions”. Misconceptions such as “all fractions are less than one” or “fractions are out of 100” were significantly fewer than we had anticipated, but they were present in both attainment groups.

There were differences in what pupils were recalling in the two lessons. One class were mainly remembering knowledge about types and actions on fractions whereas the other class were remembering fraction facts and demonstrating the minor misconceptions which we had anticipated. There was little difference in the quality of the responses between the two groups and this was irrespective of whether they were recalling facts or actions that can be performed on fractions. The table 5.2.2b below shows the categories of the responses observed from the two classes for this activity and the frequency, percentage and total responses for this category.

Categories	Number of responses	Percentage
Misconceptions	11	8%
Types of fractions	59	43%
Recalling Facts	20	19%
Numerator / Denominator	22	16%
Actions on Fractions	17	12%
None of the above	6	4%

Table 5.2.2b Categories of response by frequency and percentages for the activity learning episode of the lesson from both classes.

Analysing the written language used by pupils might help us to understand their prior learning. For example when recalling types of fractions pupils were writing simple statements relating to equivalency of fractions

Equivalent to $\frac{2}{4}$ is $\frac{1}{2}$

$\frac{1}{4}$ is the same as $\frac{2}{8}$

as well as more extensive explanations and definitions, such as

> "When the numerator is larger than the denominator it is called an improper fraction".
>
> "There are top heavy fractions, this is when the top numbers is larger than the bottom number eg 10/5"

A possible explanation for the similarity of wording and phrasing relating to equivalency of fractions observed on a number of scripts across the two classes might be the fact that the majority (34 out of 42) of the pupils had attended one feeder primary school.

The misconceptions from the lessons were

Make things fair / even
Can't go past the lowest point (2 pairs of pupils)
Out of 100 (3 pairs of pupils) *
A portion of something
All fractions are less than 1 *
Denominator is higher *
Most things can be cut into fractions
The numerator has to be smaller than the denominator *

Those starred above we had anticipated but we had expected to see other major misconceptions such as

$\frac{3}{4}$ is always more than $\frac{1}{2}$ indicating pupils were not aware or did not realise that fractions can be operators

$\frac{1}{2}$ is the same as 0.2

$\frac{3}{4} + \frac{1}{2}$ is $\frac{4}{6}$

None of these were evident, and after a short discussion with the primary school teacher, it became evident that the primary school had restricted their mathematics curriculum to fraction facts (such as knowing terms numerator and denominator), equivalent fractions and the equivalency between fractions, decimals and percentages. A number of reasons for the selection of this aspect of their curriculum were given by the primary school. The most informative comment made related to the level of mathematical expertise, knowledge and confidence of their teaching staff.

The pupils' misconceptions were shared with the teaching staff from both the primary and secondary schools. Both groups of teachers agreed that these were the types of misconceptions children arrived with from home and everyday societal experiences. The comment "makes things fair" from one pair of pupils was followed up after the lesson in a short informal discussion. From notes made at the time the two pupil pairs explained that

Me :	In today's lesson on the activity sheet what do we know about fractions you wrote "makes things fair". Can you tell me what you meant?
Pupil Pair A:	Well you know when you share something we always have the same amount it makes things fair ; then you always get the same fraction.
Pupil Pair B:	Because when you share something you all have to have the same amount of fraction it's what makes it fair.

These two pupil pairs had obviously a notion and understanding of a fraction being related to division and equitable sharing. This is often the methodology advocated in the early stages for the teaching of fractions, however they exhibited a weaker understanding of fractions as an extension of the integer number system.

The activity episode of the lesson did achieve its objectives of engaging pupils as corroborated by their teachers (appendix 12, lines 712-723) and allowing the teacher to diagnose what pupils knew about fractions as well as providing evidence of their misconceptions. Whilst the pupil outcomes from the activity can be seen, evaluated and the differences noted in the categories and types of responses above, the impact from the activity on an individuals' learning at the start of lesson was not a design feature and therefore not evaluated. However, evidence from the videos, in respect to increased frequency of pupils wanting to answer questions, does seem to suggest that the impact of pupils having initially gained a degree of success, and being actively involved in recalling learning, contributed to them being more predisposed to volunteer answers (both correct and incorrect) in the remainder of the lesson. This increase in volunteering of answers could also be attributable to a number of additional factors such as the presence of cameras and a different teacher delivering the lesson. Alternatively I would argue it could be the result of the style and design of the lesson. Nevertheless, this level of active involvement and participation from the majority of pupils in both classes had not been present in the lessons I had observed where a starter had been of a more traditional solitary, silent, exercise or the practising of a skill. It is difficult to assess the level of learning as the activity was designed to recall prior knowledge, but it does appear that pupil dialogue certainly leads to a more collaborative participatory learning environment when compared with previously observed lesson beginnings.

5.2.3 Student learning from the lesson - (skills)

The second learning episode skills (appendices 4 and 25), in each of the two lessons, was approximately 10 minutes in length. The learning episode was designed to introduce the concept of fraction division through the use of the equivalence of fractions rather than a ‘flip and multiply’ procedure. Again pupils were actively encouraged to work, discuss and construct their learning in pairs. The learning episode was in two parts comprising of a short whole class demonstration by the teacher followed by a skills practice worksheet. The initial discussion and demonstration (using the manipulative cards) consisted of a number of examples of the type:

“a half comprises of two quarters or four eights”

This was then followed by pupils working to answer a skills worksheet of just four questions which were randomly chosen by me. Pupils were then encouraged to continue the skills worksheet by designing and solving questions of their own making. The four questions set on the skills worksheet are of a standard introductory nature. These four questions can be found in most school mathematics textbooks where the numerators of the fractions are unity. However, the approach taken in the teaching was to look at how many of one fraction is equivalent to a second fraction. This is a markedly different approach from the normal almost universally used algorithmic approach of ‘flip and multiply’.

From the findings nineteen of the twenty-one pupil pairs correctly completed the four questions set, the remaining two pairs correctly completed three questions. A further thirty-two questions were devised by pupil pairs and correctly solved. An additional two challenge questions needed to be set by me for two pairs of pupils.

The two further questions set by me were:

$$\frac{2}{3} \div \frac{4}{6} \qquad\qquad \frac{1}{3} \div \frac{2}{9}$$

and both of these were answered correctly by both pupil pairs with no interventions from any teacher.

Of the initial questions set only three solutions out of a total of eighty-four were incorrectly answered (all from pupil pairs in the lower attaining class). This was a surprise both for the teachers and me. Normally pupils struggle with this mathematical operation on fractions and make numerous errors when trying to apply the traditional ‘flip and multiply’ method. Pupils constructed a further thirty-two questions not always following the exemplar worksheet questions where one denominator was always a multiple of the other. Five questions written and correctly solved by pupils involved numerators that were not unity.

Examining the sequence of some of the additional eighty-four questions produced by pupils might give an indication of their thought processes, for example one pair of pupils produced the following sequence of questions

<u>Example set 1</u>

How many $\frac{1}{64}$s are there in an $\frac{1}{8}$?
How many $\frac{1}{128}$s are there in an $\frac{1}{8}$?
How many $\frac{1}{256}$s are there in an $\frac{1}{8}$?

A second pair of pupils produced the following sequence of questions

<u>Example set 2</u>

How many $\frac{1}{64}$s are there in an $\frac{1}{8}$?
How many $\frac{1}{9}$s are there in a $\frac{1}{3}$?

How many $\frac{1}{300}$s are there in a $\frac{1}{5}$?

It would appear from these two representative sets of examples that the first pair of pupils was using the well-rehearsed doubling strategy to generate questions. From the video this pair of pupils had also noted the implication of the doubling strategy on the pattern in the solutions, with one pupil saying "if we double the denominator the answer is twice as big".

It is also evident that the second pair of pupils were experimenting and exploring other relationships between denominators, with the first two examples being accounted for by the fact that pupils had studied square numbers immediately prior to the research lessons, indicating that these pupils are transferring or linking mathematics across topics. The third question (their final question) is more difficult to explain and there was no indication on any of the videos about how pupils had arrived at the questions. More importantly there was no indication as to how they had correctly solved their own question.

It had been anticipated that pupils would use the doubling strategy because of their familiarity with the technique from both their primary school and their first year in secondary school mathematics lessons. The second set of examples above had not been anticipated as previous observations had given no indications that this might occur. An inference might be that the design of this skills learning episode is flexible enough to support learners at whatever stage they are at in their conceptual development, whilst allowing pupils the independence and freedom to explore mathematics at their own level.

Additionally, the variety of the first questions generated by the pairs of pupils was interesting

How many $\frac{1}{12}$s are there in a $\frac{1}{6}$?
How many $\frac{1}{64}$s are there in a $\frac{1}{8}$?
How many $\frac{1}{9}$s are there in a $\frac{1}{3}$?
How many $\frac{1}{15}$s are there in $\frac{1}{3}$?
How many $\frac{1}{200}$s are there in a $\frac{1}{50}$?
How many $\frac{1}{8}$s are there in a $\frac{1}{40}$?
How many $\frac{1}{35}$s are there in a $\frac{1}{5}$?
How many $\frac{2}{6}$s are there in a $\frac{2}{3}$?

With the exception of the last question it would appear that pupils were initially following a strategy of producing questions where one denominator was a multiple of the other. This was what had been presented in the worksheet questions, so the inference was they were simple copying a pattern to generate their questions. However, the creativity demonstrated by the pupil pair in the design of the last question above could not be catered for in a more traditional textbook or worksheet questions.

From the video evidence every pair of pupils used the manipulative cards to solve the initial four questions. Those pairs of pupils that began to explore other relationships (ie not using a doubling strategy) became less reliant on the manipulative cards. The two pupil pairs who needed to be set more challenging questions did not rely on the manipulatives at all. This would tend to suggest that pupils need the support of visual or physical aids to cement their understanding of concepts before being able to move onto the pure algorithmic methods or more abstract conceptions and this is the complete opposite of the normal approach taken when teaching this topic. The two photographs (extracted from the video data in 7AC) indicate the use of the manipulatives for making comparisons with the first demonstrating pupils solving how many $\frac{1}{12}$s are there in a $\frac{1}{3}$ and a whole, this is in construction, ie the pupil had not completed it when the photograph was taken.

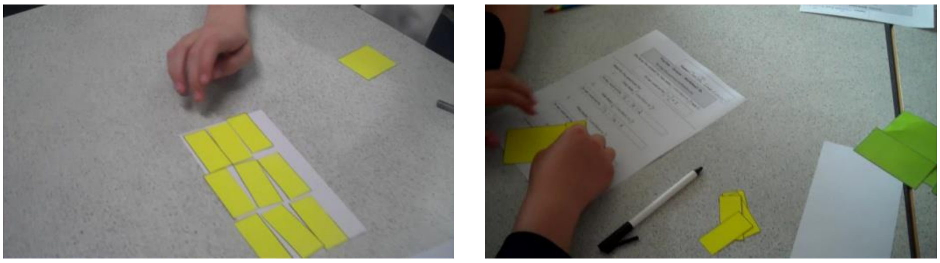


Figure 5.2.3 Photographs from the study lesson 7AC

After an initial period of time the majority of pupils in one class discarded the manipulative tiles, whereas the other class continued to use them for the whole of this learning episode. The consequence was that the former group were not distracted, nor did they need to rely on, the manipulatives in order to devise and correctly solve significantly more questions than the latter group. Whilst the quantity of questions devised and solved is not being regarded as an indicator of learning, the complexity of the questions generated does, and can be taken as a measure of deeper conceptual learning.

The conceptual difficulty of the mathematical skill of dividing fractions should not be viewed in terms of the sheer quantity of work generated, in a relatively small time period, nor should the reliance on the manipulatives. However, a single factor that might account for the differences in learning outcomes between pupil pairs might be the conception of division in terms of the phrase “how many”. Again this is not the normal approach taken by teachers when dealing with the division of fractions. It was certainly not the approach that either class had experienced in the first year of their secondary mathematics education when dealing with division of integers or decimals.

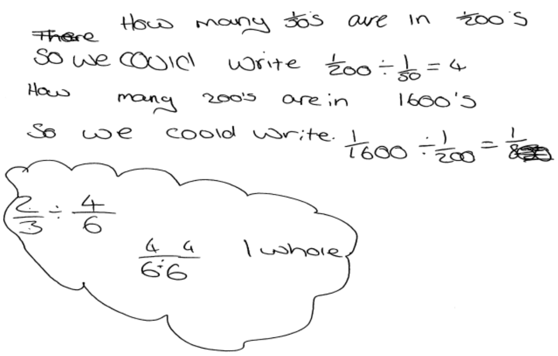
Looking a little more deeply into two responses where the pupil pair had devised the questions may help to illuminate how this skills learning episode supported the pupils’ thinking processes. The final question created by one pair of pupils, who had already correctly solved seven questions, was $\frac{1}{250} \div \frac{1}{500} = 2$. All of their previously designed questions had followed exactly the same doubling strategy. This seems to indicate that these two pupils might have seen a pattern and were simply following the pattern rather than demonstrating any thinking processes or any levels of deeper understanding.

Much as the first pair of pupils above the second pair of pupils followed a doubling strategy for the first two of their final three questions.

$$\frac{1}{50} \div \frac{1}{200} = 4 \qquad , \qquad \frac{1}{1600} \div \frac{1}{200} = \frac{1}{8}.$$

However their final question was

$$\frac{2}{3} \div \frac{4}{6} = \frac{4}{6} \div \frac{4}{6}$$
$$= 1$$



which was completely unconnected to ones that immediately preceded it. Looking at the video of this pair of pupils they were generating and finding solutions where one pupil was challenging the other to make ever more difficult questions for the other to solve:

Pupil A	If we divide $\frac{1}{100} \div \frac{1}{200}$ we get 2.
Pupil B:	So if we $\frac{1}{200} \div \frac{1}{400}$ we get 2,
Pupil A:	Yes, that's right because 400 shared by 200 = 2
Pupils	After a little more discussion relating to doubling they turn to the pair next to them, who appear to have been listening and replicating the fractions generated by the first pair, and begin to work as a four They agree to work on the generating solutions to the fractions so that the answer is 2

$$\frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}, \frac{1}{96}, \frac{1}{192} \dots \dots \dots ..$$

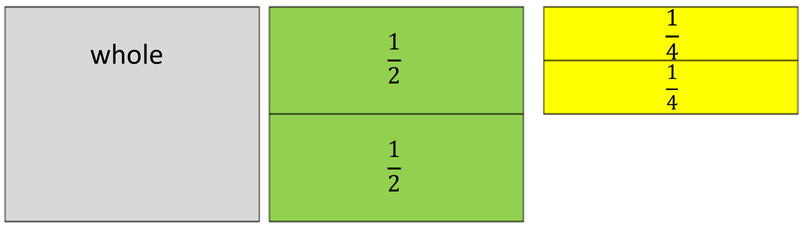
With each pupil checking to make sure the pattern is followed ... they eventually end at

$$\frac{1}{1536} \div \frac{1}{3072} = 2$$

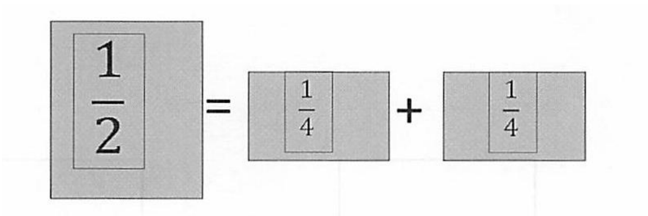
During this interchange between pupils the video shows there were no interventions or conversations with teachers. What began as a development of a self-contained skill, dependant on the use of the manipulative tiles, eventually allowed the higher attaining pupils to extend their learning through some self-directed challenge and creative inquiry. The same impact was not so noticeable in the lower attaining group. Factors that could account for the discrepancy between the groups are pupil motivation, levels of engagement with the mathematics or the spending of unproductive periods of time with the manipulatives. However, there was no evidence of any of these in the videos, on the pupil answer scripts, comments from the participating teachers or from written thoughts provided by the pupils at the end of the lesson. I concluded that the conceptual difficulty of both the skill and the requirement from pupils to construct their own questions to solve were the two discriminating factors that accounted for the differences between the attainments.

5.2.4 Student learning from the lesson – (Exercise)

The third learning episode exercise (appendices 5 and 26) of each of the two lessons was designed to reinforce the concept that a fraction is able to be made from combining two smaller fractions for example $\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$. Whilst this is not the division of fractions the concept of being able to replace one fraction with two smaller ones was required for the final episode of the lesson where pupils would need to replace $\frac{3}{4}$ with $\frac{3}{8} + \frac{3}{8}$. Again, in this part of the lesson pupils were able to support their understanding by using the manipulative tiles to reveal that a tile can be replaced by combinations of other tiles, for example:



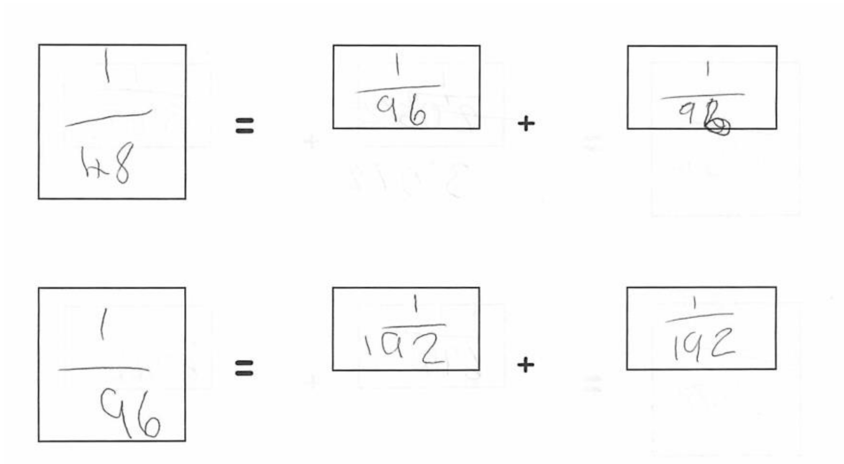
After some initial introductions, from me, on how to move from the practical visualisation of fractions in the form of the manipulative tiles, to the more formalised symbolic representation of the addition of two fractions



an open-ended exercise was given to pupils. The move from fractions being represented by tiles to being represented by symbols created a number of questions from pupils. The questions from pupils mainly related to how to record the fractions rather than the meaning of the symbols. The open-ended exercise had no prescribed questions just the single example above and pupils were again required to generate their own questions and solve them. Having gained the skill from the previous learning episode a key feature or component of an exercise is that of the freedom and legitimacy for pupils to explore the mathematics. This freedom is the complete opposite to the normal textbook or worksheet prescribed set of questions.

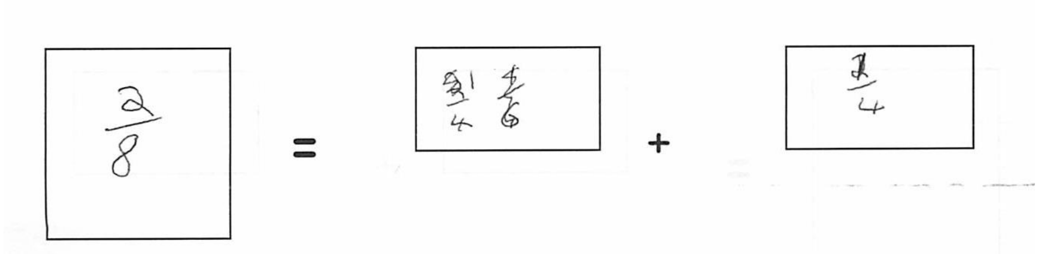
A total of 135 questions were created with both classes producing exactly the same percentage of correct solutions (82%). This might indicate that the conceptual difficulty for pupils to create their own questions was not dependant on their mathematical attainment but was an outcome of the lesson design and the two prior learning episodes.

The findings demonstrated two distinct categories of outcomes. In the first category pupils were again using a doubling strategy to effectively generate questions. The doubling strategy seemed to support the symbolisation in that pupils were clearly able to see the connections in the denominators. For example, one of the pairs of pupils, both of whom were actively involved in constructing seventeen correct questions and solutions in just 8 minutes, relied on the doubling strategy that had been developed in the preceding lesson episodes (skills)



5.2.4a Pupils demonstrating conceptual understanding

The second category used a variety of random strategies with no clear logical approach to the creation of questions. A common factor to this category was that the symbolisation and recording of the question and solution proved to be a difficulty which was not the case in the other category. A different pupil pair, again both of whom were actively involved in constructing three correct questions and solutions in the 8 minutes, generated a set of random questions but used the manipulatives to correctly demonstrate their conceptual understanding of how two identical fractions can be combined (evidenced in the video footage and in table 5.2.4c below). They found the symbolisation to be much more difficult, as can be seen in the errors and corrections made in the recording of the solution.



5.2.4b Pupils demonstrating conceptual understanding

At the outset of the exercise a total of just nine incorrect solutions were generated by all of the pairs of pupils. Once I had intervened no further incorrect solutions were seen from these pupil pairs. These nine incorrect solutions can again be divided into two distinct categories, the first category being a misconception of how to apply the doubling strategy for example

$$\frac{1}{12} = \frac{1}{6} + \frac{1}{6}; \quad \frac{1}{24} = \frac{1}{12} + \frac{1}{12};$$

$$\frac{1}{48} = \frac{1}{24} + \frac{1}{24}; \quad \frac{1}{96} = \frac{1}{48} + \frac{1}{48} \text{ and } \frac{1}{4} = \frac{1}{2} + \frac{1}{2}$$

The second category with no clear rationale contained just four questions with incorrect solutions

$$\frac{1}{3} = \frac{1}{2} + \frac{1}{3}; \qquad 1 = \frac{1}{3} + \frac{1}{2}; \qquad 1 = \frac{1}{6} + \frac{1}{6}$$

$$\frac{2}{4} = \frac{4}{12} + \frac{4}{12}$$

Evidence from the video was that pupils were misaligning tiles or pupils leaving small gaps between the tiles. The final question and incorrect solution was attributable to a simple error in perceiving the size of the equivalent manipulative tile for $\frac{4}{12}$, ie it was $\frac{1}{3}$ and not $\frac{1}{4}$ and hence the pair would have achieved the correct solution of $\frac{2}{3}$ rather than $\frac{2}{4}$. Here the manipulatives had been a hindrance rather than an aid.

The errors in the first three questions above are less easily explained and the video and written solutions provide little clues as to the ways pupils arrived at these answers. Two further interesting questions of note were devised by pupils

Question 1: $\frac{1}{3} = \frac{2}{12} + \frac{2}{12}$

Question 2: $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$

In the first question pupils initially replaced $\frac{1}{3}$ with $\frac{1}{6} + \frac{1}{6}$ and then replaced $\frac{1}{6}$ with $\frac{2}{12}$ a double substitution. This concept had not been discussed or even suggested, but the video footage of the pupils working on this question captured part of their thinking as indicated in the extract below:

	Pupils talking	My commentary
Pupil a:	Let's do one third	
Pupil b:	OK, you find the two that makes one third	Pupil B actually picks up the $\frac{1}{12}$ card rather than $\frac{1}{6}$ card by chance. There is some discussion partially inaudible about not being able to get 2 cards of $\frac{1}{12}$ to make $\frac{1}{3}$.
Pupil a:	That can't be right as we need more than 2 of them. You got it wrong. What about this card?	Pupil A picks up the $\frac{1}{6}$ card
Pupil b:	Yea, that works 2 of them makes this one *(referring to the $\frac{1}{3}$ card)*	
Pupil a:	Look two of them make this one *(The pupil refers to the $\frac{1}{12}$ and the $\frac{1}{6}$ cards)*	There is then a break in the footage but on the desk we can see cards representing $\frac{1}{3} = \frac{1}{6} + \frac{1}{6} = \frac{2}{12} + \frac{2}{12}$

Table 5.2.4c Pupils demonstrating conceptual understanding

These two pupils are beginning to explore fractions, build connections between fractions and this might be due to the open – ended nature of the learning episode where pupils have the freedom to both construct and build questions that they consider to be interesting. These freedoms in standard textbook or worksheet exercises are not normally seen, however, it might be argued that this outcome was due to pure chance. By creating the conditions for experimentation and discovery, I would argue that such outcomes are more likely to happen using this approach than if pupils are merely presented with a closed set of questions from a textbook.

The pupil devised question $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$ was the only one produced by any pairing in either class where the decomposition resulted in two fractions with different denominators. No video footage existed of this, but the sequence of questions on the pupil sheets immediately prior to this one were:

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{4}; \qquad \frac{1}{4} = \frac{1}{8} + \frac{1}{8} \qquad \text{and } \frac{1}{3} = \frac{1}{6} + \frac{1}{6}$$

If we were expecting to see a question of this type (unequal denominators) then, from experience, we may have expected to firstly see $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$; but given we were asking pupils to replace one card with the combination of two others and the fact that no card existed for $\frac{3}{4}$ it might be unreasonable to assume that pupils would generate the more expected question.

Studies by Carpenter and Moser (1982), Steffe, Thompson and Richards(1982) and Hartshorn and Boren (1990) have concluded that pupils benefit when using manipulatives tiles during problem solving when moving from intuitive to logical thinking and from the concrete to the abstract. This study would support these findings. Indeed the common misconception that $\frac{2}{2a} = \frac{1}{a} + \frac{1}{a}$ did not appear in any of the solutions to the questions posed by pupils after the initial teacher intervention, I would argue that this is almost certainly attributable to them being able to manipulate and count the resulting fractions tiles. However, I would caution also that manipulative tiles can cause cognitive conflicts especially when trying to move from the concrete to the abstract written symbolisation of fractions as in the examples quoted above.

In all of the video footage studied it appeared that pupils were less frequently using or reliant upon the manipulative tiles than in the previous learning episode (skills). Only 12 out of 21 pupil pairs used the tiles in this episode as compared with all pupil pairs in the skills episode. This tends to indicate that there might be a learning progression of skills followed by exercises that could be promoted. The instrumental learning (Skemp, 1978) developed and promoted by the previous skills learning episode of the lesson was then applied and developed during this exercise episode. This is what Skemp (1978) calls ‘becoming relational’. Those pupils, the majority, who produced more than 5 questions, relied upon the support of the cards less frequently than the pairings who produced and solved fewer questions. This might tend to suggest that decomposing fractions is less conceptually difficult than finding how many $\frac{1}{16}$s there are in $\frac{1}{8}$? as developed in the skills episode. However, the lack of use of the manipulative tiles could also arise from the pupil’s familiarity with the addition of fractions with common denominators. Simple examples of the addition of fractions with common denominators had been part of the primary school mathematics curriculum. It could also be as a direct consequence of the learning episodes presented in the lesson, and the fact that pupils were becoming more confident about their understanding of fractions.

Again the sheer quantity of work produced in a relatively small period of time surprised the participating teachers, as did the quality of the pupil discussions when the video footage was viewed. During the course of the discussions one teacher noted that the normal practice for the school was to make pupil pairings of different genders. This whole school strategy was adopted to minimise behaviour management and increase the quantity of work produced. In these two lessons pupils were able to work in friendship pairs but it was not a deliberate decision or part of the research design. It was rather part of my normal classroom practice that I have adopted over my teaching career. During the discussion immediately after the lesson with two teachers they noted that the quantity of work and discussions produced were possibly as a result of the non-compliance to the normal practice of pairing different genders:

Me: What were your impressions of the quantity of work produced?

Teacher H: "I think it was far greater than normal. Also if you had put them 'boy-girl' then that seating might not have worked so well. I also don't think they were sat in their normal seating plan, simply because we moved some of them to make pupil pairs. I think if you had put them in a 'boy-girl' pair they wouldn't have done as much discussing".

Teacher D: I hadn't anticipated the quantity of work that they would produce as I thought they would struggle with this exercise. That wasn't the case.

Me: Why do you think they produced so much?

Teacher H: They were obviously interested in the work. The cards made a difference. I suppose the biggest difference might have been that they were making up the questions and answering them – they had ownership and the pairings definitely influenced the discussions and the quantity of work produced.

Me: What made you think they would struggle?

Teacher D: Just knowing the group. Perhaps I underestimate what they are able to do. Perhaps our boy-girl behaviour strategy needs to be revisited.

This final comment is an interesting reflective observation from a teacher who had only recently qualified. It might indicate that teacher attitudes and beliefs are formed about classes very quickly once they begin their professional life. It is worrying that this teacher is attributing the success to an organisational change rather than a pedagogical change, but this might simply be due to inexperience. Additionally the findings from the exercise episode of the lesson would seem to suggest that compared with the norm there is an increase in quantity of work produced with fewer misconceptions or mathematical errors. Also there appears to be a hierarchy between exercises and skills, where exercises can and should build on the skills already learnt.

The distinction between a skill and an exercise is therefore very subtle. This research would venture that a skill (in the form used here) has the distinctive feature of :

> the manipulation of mathematical objects to find relationships that aid simplification. (In this research the skill developed was the replacing of one fraction by two or more smaller fractions). Skill development is often the first step in the conceptualisation and the solving of problems

whereas

> an exercise is the application of a skill combined with a mathematical operation which tends to lead to an overall deeper conceptualisation of the mathematics. (In this research the exercise developed the concept that one fraction is the combination of two or more smaller fractions with the operation of addition) Watson, Jones and Pratt (2013, p.28).

It is these two conceptualisations of a skill and an exercise that makes a qualitative difference between the two learning episodes, and consequently the pedagogical understanding of the two terms.

Underpinning all mathematics is the notion that mathematical objects (here in the form of fractions) can be combined using operations (here addition) to simplify or generate other mathematical objects. The skill of manipulating the objects creates the foundation for exercises and the knowledge of how to combine the objects to create deeper conceptual mathematical understandings.

5.2.5 Pupil learning from the lesson – (Task)

The final learning episode of each of the lessons was a task (appendix 6). It was designed to support the learning of the division of fractions using a familiar, real world context and to allow the pupils to apply their learning from the previous episodes to a specific problem. The difference between the task and the precious two episodes (skills and exercise) is that the task has its foundation in a real world problem. Here the pupil has to apply a range of mathematical skills and knowledge, previously gained, to a real problem rather than an artificial situation. Supporting pupil understanding of real world problems through the use of objects is well reported in the literature Cockcroft (1982), Adhami, Johnson and Shaver (1998) and Smith (2004).

The task in this research was designed to build on the knowledge and understanding gained in the previous three episodes of the lesson. Additionally it aimed to support the cognitive development of the concept of the division of two fractions, the visualisation of fractions and the idea that a division operation can be conceptually reformulated using the phrase “how many”. The task was grounded in the thinking of Adhami, Johnson and Shaver (1998) where challenges or tasks within a context (such as finding the area of a trapezium by considering house roofs) are preferred to the more conventional instructional didactic teaching approaches.

The learning episode (task) required the pupils to solve the problem

> “A bottle has $1\frac{3}{4}$ litres of concentrated squash. In each glass the party host wants just $\frac{1}{8}$ litre of the concentrate before topping the glass up with water. The problem is the host needs to find out how may glasses of squash she can make from one bottle of the concentrate. Can you help?”

The decision not to use water and squash was taken due to a number of medical concerns relating to the contents of the squash. So it was anticipated that pupils would rely heavily on the manipulative cards as a substitute for the water and squash. The video evidence from both classes shows that the manipulative tiles were used by all pupil parings, even though there were no direct instructions from me to do so, to model and demonstrate the number of $\frac{1}{8}s$ in $1\frac{3}{4}$.

Pupils needed to use the concepts developed in the exercise and skills learning episodes of the lesson to replace the mixed number $1\frac{3}{4}$ with $\frac{7}{4}$'s and then $\frac{14}{8}s$ so that they were able to calculate how many $\frac{1}{8}s$ they would require. No introduction or explanation was given by me other than just reading through the problem for the pupils.

Prior to this task neither the skills nor exercise phase of the lesson had involved mixed numbers. Pupils were therefore being asked to make a conceptual jump from finding either

> How many $\frac{1}{64}s$ are there in an $\frac{1}{8}$?

or

> demonstrate that a fraction is able to be made by combining two smaller fractions to demonstrate the concept that $\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$.

to find how many $\frac{1}{8}s$ are there in an $1\frac{3}{4}$. I was unable to find any instances on the video where this caused any pupils a concern, they were all able to model the problem immediately and replace unity with four $\frac{1}{4}s$. It is easy to theorise that their capacity to make this conceptual jump was due to mathematics being either

1. framed in a real world task
2. or that the manipulative tiles supported the understanding
3. or that working as pairs, collaboratively to get a solution

but finding evidence in the video and the work produced by the pupils in the lessons to support these speculations proved to be difficult. However, the written reflections from pupils (appendices 28 and 29) would tend to support these speculations as evidence in the reflections from pupil:

1. It was actually really fun, easy, interesting and challenging
2. It was easy because of the tiles
3. It was easy working with my partner

In total there were 17 solutions to the original task from the 21 pupil pairs and a further 17 pupil designed tasks with solutions. In total out of the 38 solutions there were only 9 incorrect solutions produced. Analysing the three incorrect solutions produced to the original problem by pupils in one class they had all made the same error from the same misconception. Each pairing had correctly calculated that $\frac{3}{4}$ was $\frac{6}{8}$ and then incorrectly doubled the $\frac{6}{8}s$ to give $\frac{12}{8}s$ instead of replacing unity with $\frac{8}{8}$ and adding $\frac{6}{8}$ and $\frac{8}{8}$ to get the required solution. These three pupil pairs were using the strategy developed in the skills episode where they had been systematically doubling the previous fractions. I looked at the video evidence for clues as to the reasoning employed by these pairs. All three pairs had used the manipulative cards to correctly achieve the equivalence of $\frac{3}{4}$ and $\frac{6}{8}$ but there was no evidence that any pair had used the manipulative cards to replace unity with $\frac{8}{8}$.

From past experience with other classes using the normal algorithmic methods this example of a misconception was expected and could have been predicted. In discussions prior to the lessons the participating teachers had identified that this misconception might occur in all of the learning episodes, however it was only seen in this final part of the lesson.

Three of the incorrect solutions were produced from questions written by pupils themselves after they had correctly completed the original task. Two incorrect solutions are difficult to analyse as pupils just produced an answer with no indication of working or method. However the third pair wrote the following question:

> A bottle has $6\frac{3}{4}\ litres$ of concentrated squash. In each glass the party host wants just $\frac{1}{8}litres$ of the concentrate before topping the glass up with water.

then correctly worked out that 6 litres would require 48 glasses of $\frac{1}{8}$ concentrate and added the further 6 glasses for the $\frac{3}{4}$ but wrongly totalled the answer to 52 rather that the correct answer of 54. This error is just that; an error; and there was no misconception in the thinking. No pupil wrote a question with a different context and all of their questions were framed in the original problem context. Given this was the first time they had met the topic and being allowed the freedoms to create their own questions, the conceptual difficulty of the mathematics, and the fact that I was a 'new' teacher to them, I suppose this is understandable.

In conversations immediately after the lessons teachers were surprised by the quantity of correct solutions and additionally had thought that the context would create major difficulties for pupils as they would not be able to identify the mathematics in the problem. Again the solutions were produced in a relatively limited period of time which surprised the teachers with one teacher commenting that

> "It was a good lesson, pupils produced significantly more work than normal and appeared to understand what was going on".

The telling phrase "appeared to understand" from this teacher would seem to indicate a level of pupil understanding from the task that she was not expecting. The same teacher was not convinced that the process was equivalent to the more normal "flip and multiply" method, commenting that

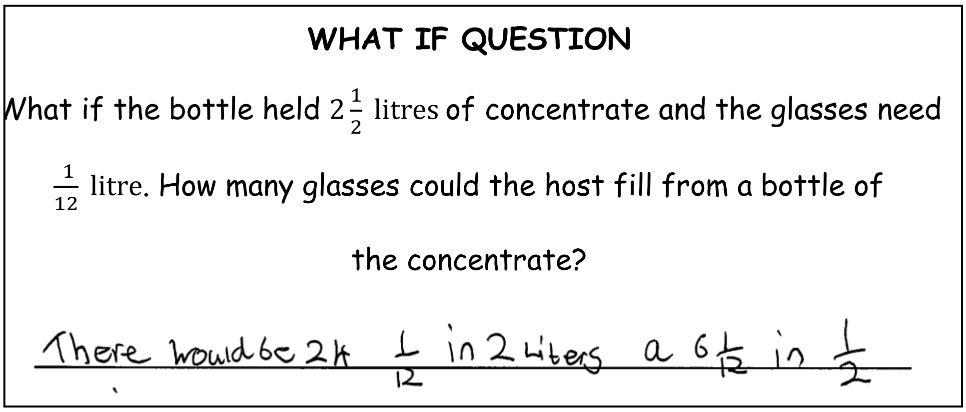
> "They did apply the learning to a task at the end, but it comes back to whether they were dividing. [Pupil X] split up the $1\frac{3}{4}$ into $\frac{8}{8}$ and the $\frac{3}{4}$ into $\frac{6}{8}$, eventually getting $\frac{14}{8}$. So he has done an addition of the number of eights".

But the same teacher also commented

> As far as they were concerned they were seeing how many times this fraction fits into another fraction.

This teacher seems to be saying that the pupil perceives the problem in terms of the addition of fractions. I would totally disagree with this viewpoint. The pupil has quite correctly perceived the problem in terms of how many $\frac{1}{8}s$ will divide into $1\frac{3}{4}$, with the requirement to find an appropriate representation for $1\frac{3}{4}$ in terms of $\frac{1}{8}s$ before solving the problem. I am firmly of the belief that all forms of division should be taught using the phrase "how many" (Ball and Wilson, 1990; Matthews, 2014). Pupils start the process of dividing integers by being asked for example how many twos are there in eight. It seems perfectly logical and mathematically sound to apply this methodology to the division of fractions. The inter-relationship between operators, and the facility to be able to select the most appropriate methodology to achieve this conceptual understanding, is at the heart of good sound teacher subject knowledge. Teachers need to be open and receptive to new methods and this was certainly the case for the teacher above who took the lesson to deliver to another class after the research had concluded.

Analysis of the response rates from the two classes (Table 5.2.5) would tend to suggest that there is little difference in either the correct or incorrect number of responses from the two differing attainment groups. This is interesting in that the teacher of the lower attaining group stated that the group would have normally found this problem inaccessible using the traditional teaching approaches of the department. Pupils from this teaching group were not only able to correctly answer the task problem but they were able to successfully demonstrate their mathematical reasoning as indicated in the script from pupil pair G.

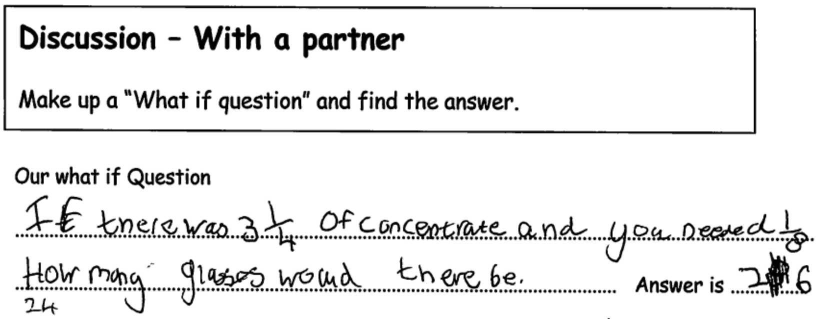

WHAT IF QUESTION

What if the bottle held $2\frac{1}{2}$ litres of concentrate and the glasses need $\frac{1}{12}$ litre. How many glasses could the host fill from a bottle of the concentrate?

There would be 24 $\frac{1}{12}$ in 2 Liters a 6 $\frac{1}{12}$ in $\frac{1}{2}$

Figure 5.2.5a Script from class 7NR – Pupil pair G.

Furthermore the same pupil pair, from the lower attaining class, were able to use the manipulatives, combine the concepts learnt and practised in the skills and exercise episodes of the lesson, in order to produce their own question and correct solution.


Discussion – With a partner

Make up a "What if question" and find the answer.

Our what if Question

IF there was $3\frac{1}{4}$ Of concentrate and you needed $\frac{1}{8}$. How many glasses would there be. Answer is 26

24

Figure 5.2.5b What if questions from class 7NR – Pupil pair G.

Pupils from the other attainment group produced slightly harder questions in the sense that the numbers were larger and the fractions were not limited to the one in the set problem but, not unexpectedly, they were then unable to solve their own questions.

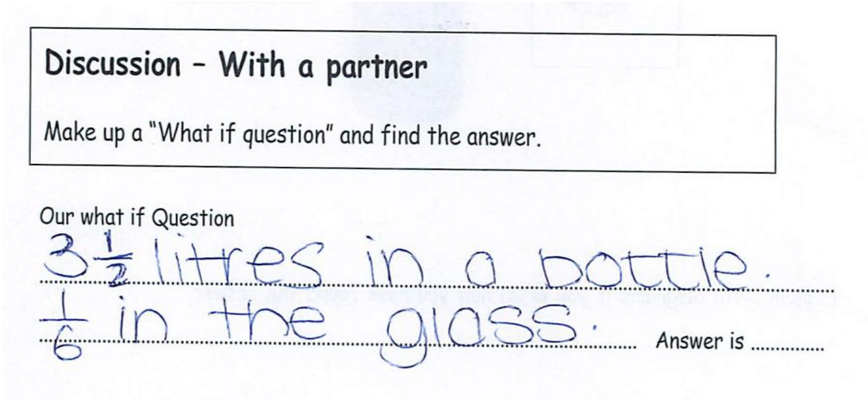


Figure 5.2.5c What if questions from class 7AC – Pupil pair B.

No pairs of pupils in either class produced a question that could not be solved based on the set of manipulatives they were given. It could therefore be argued that the real world context occupies a less important role than the support given by the manipulative tiles. For example the pack of manipulatives did not contain a card for the fraction $\frac{1}{5}$ and so no questions of the type

> "What if the bottle held $2\frac{1}{2}\,litres$ of concentrate and the glasses need $\frac{1}{5}\,litre$. How many glasses could the host fill from a bottle of the concentrate?"

were devised by pupils. However, it could be argued that the success from the other parts of the lesson (activity, exercise and skill) together with a real world task supported the pupils in the creation of 48 additional questions of a "What if " nature, of which 41 (85%) were then correctly answered (table 5.2.5).

5.2.6 What effect would a change in lesson design have on pupils' views of learning fractions?

So I designed a lesson plan (appendix 7) around four learning episodes based on the definitions (appendix 35) for an exercise, an activity a skill and a task to be used for the study. The plan details the mathematical content, the pedagogical approach and the learning intentions for each episode. The underlying teaching approach for the mathematical concept of equivalent fractions is contained in the supporting lesson presentation and the accompanying pupil worksheets (appendices 3, 4, 5, 6 and 9).

I taught the lesson plan twice adhering strictly to this plan. I then viewed the videos of each session a number of times. My analysis of the video footage only considered sequences of interactions of five or more seconds in length (appendices 33 and 34). In addition to these two lessons I had previously undertaken a shorter pilot lesson and from the resulting video I decided that I would code an interaction as either

- pupils talking together about the mathematics,
- or a single pupil engaged with the mathematics
- or as a dialogue between the teacher and pupils.

From the longer research videos a further interaction of pupils connecting learning either together or on their own became apparent. For each of the four learning episodes in the two study lessons I simply counted the number of times one of the following eight types of interactions (criteria) was observed and was 5 or more seconds in length:

A – Teacher Input – teaching / demonstrating / explaining
B – Pupil – Pupil Dialogue
C – Pupil Reasoning
D – Teacher Interventions
E – Pupils using the manipulatives
F – Pupil – Teacher Dialogue
G – Pupil demonstrating understanding
H – Connecting learning (eg division of numbers with division of fractions)

These eight criteria were selected to be as representative as possible of the aspects of learning I expected from pupils. From viewing the videos I came to conclusion about the types of pupil learning and the associated learning theory that was taking place for each of the criteria. Learning was classified as either of a surface nature or was deeper and more conceptual. There are numerous definitions in the literature for these categories of learning. I have taken the characteristics for these two types of learning to be those as defined by Biggs (1999) where surface learning is isolated and unconnected, and deep conceptual learning where concepts are being connected together.

The predominant learning theory adopted in the study lesson was that of social constructivism (see literature chapter) where pupils were required to discuss the mathematics, to work in pairs, and to share understandings so as to construct their own meanings and knowledge of the division of fractions. The table 5.2.6a below summarises and classifies the types of learning observed against each of the criteria A to H above.

Criteria	Aspects of pupil learning	Learning theory
A – Teacher Input – teaching / demonstrating / explaining	Surface Learning Passive Learning	Didactic
B – Pupil – Pupil Dialogue	Deep conceptual, active Learning	Social Constructivism
C – Pupil Reasoning	Deep conceptual learning	Constructivism
D – Teacher Interventions	Passive Learning	Didactic
E – Pupils using the manipulatives	Deep conceptual, active Learning	Experiential
F – Pupil – Teacher Dialogue	Active Learning	Social Constructivism
G – Pupil demonstrating understanding	Deep conceptual, active Learning	Constructivism
H – Connecting learning (eg division of numbers with division of fractions)	Deep conceptual learning. In the main active learning	Constructivism

Table 5.2.6 Summary of the aspects of learning for the eight criteria.

Having applied the criteria to the videos of the two study lessons I then simply counted the frequency of each criterion. All the frequencies for each of the criteria were then accumulated to give a composite view across all the footage from both study lessons. Table 5.2.6 shows the range and frequency of pupil learning (according to these eight criteria) in each of the four episodes of the lesson (ie activity, skill, exercise and task).

The data in the table 5.2.6 would appear to suggest that maximum number of interactions against the majority of the criteria (A to H) tends to occur when the learning episode is based on a task, with the inference for lesson design and consequently the method by which pupils prefer to learn. The embedded mathematical context in the task appears to be an important element for pupils' learning especially connecting learning (criteria H) and reasoning (Criteria C) as these two criteria are most prevalent when pupils were engaged with the task part of the lesson.

I was surprised by the results for the criteria E (the use of the manipulatives – the tiles that could be physically moved); because I had envisaged that they would be simply a resource that would engage and motivate pupils in all four learning episodes. This had certainly been the case in the piloting of the materials. But as the table 5.2.6 shows manipulatives featured significantly when pupils were doing either a skill or the task. I had expected that manipulatives would have been principally used when learning a new skill or answering an exercise to support the conceptual understanding of a fraction being the composite of two smaller fractions. Shaw (2002, p. 2) had indicated that “manipulatives and models are valuable resource tools for engaging students in the language and communication of mathematical ideas and concepts”.

In the NRC (2001, p. 354) publication it is argued that “manipulatives can provide valuable support for student learning when teachers interact over time with the students to help them build links between the object, the symbol, and the mathematical idea both represent”. The publication also suggests that the use of concrete manipulatives underpins conceptual understanding and this is particularly so for low attaining pupils. This view was supported by both Marsh and Cooke (1996) and Ruzic and O’Connell (2001) separately concluding that using manipulatives is especially useful for teaching low achievers, and pupils with learning disabilities. They all reasoned that manipulatives helped facilitate pupils and teachers discussions and assisted pupils in reflecting on the conceptual understanding of the mathematics being taught. Whilst discussion and mathematical reflection are important for all pupils they are often not commonly observed in low attaining pupils (Tereshchenko et al., 2018).

This research prompted me to further explore the extent to which pupils in the one class used manipulatives. Chart 5.2.6a summarises all the interactions for this lower attaining class (7NR) and this clearly shows for this teaching group manipulatives were used most frequently when engaging with a skill or an exercise. This supports the research of Marsh and Cooke (1996) and Ruzic and O’Connell (2001).

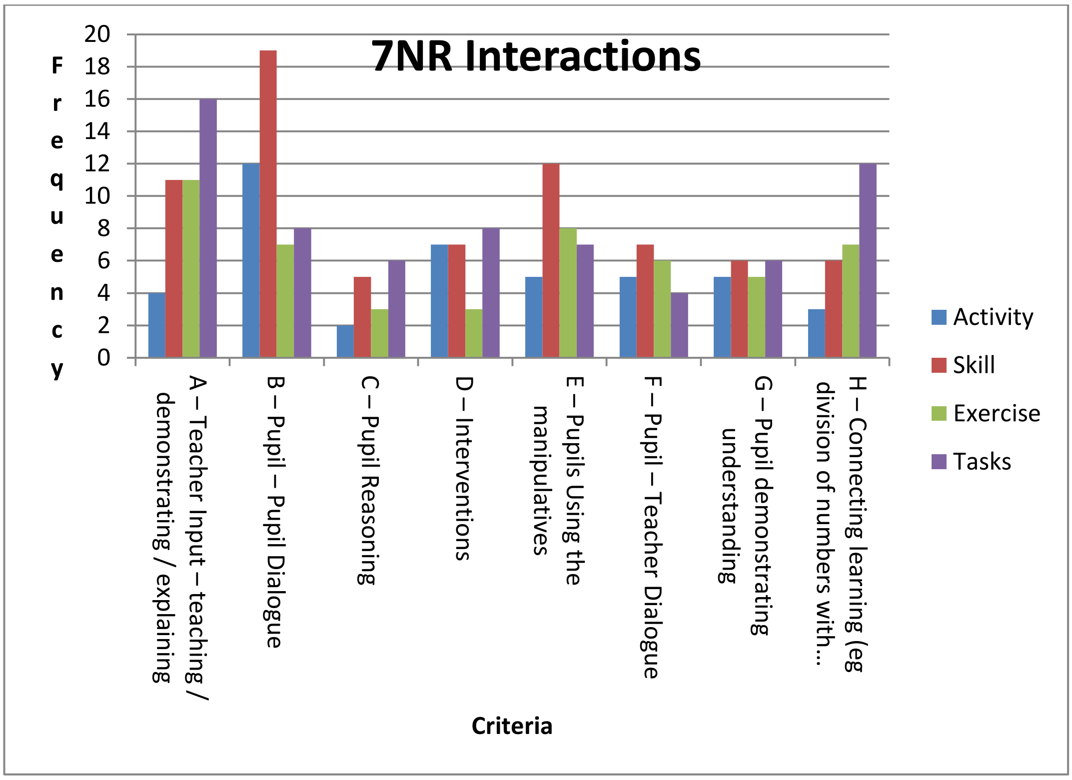


Chart 5.2.6a summarises all interactions for the class 7NR

This would tend to indicate that the use of manipulatives to support learning is important when new skills or exercises are introduced to classes, and this is especially so here where their conceptual understanding of the relationships and mathematical operations on fractions are just beginning to be formed. Pupils in the class expressed a range of views about the manipulatives:

Pupil A: They helped me understand fractions more, and they are easy to use.

Pupil B: They helped me understand because I could see them and help me understand it is better this way I think

Pupil C: They helped me to see how many fractions were in a fraction

In general the class was of the opinion that the lesson had been made easier by them being able to use the manipulatives.

In the case of the second class (7AC) where the conceptual understanding of fractions was well-formed the findings indicated that the pupil used the manipulatives predominantly for solving the task. However, Chart 5.2.6b also indicates that for the real-life contextual task produced maximum frequencies in seven of the eight criterion (A – H). This might imply that pupils in this class, operating at a conceptual higher level than those in the other class, found the task to be more engaging. They also were able to apply the mathematical concept to the task even though they were still using the manipulatives as a support mechanism.

> "A bottle has 1 $\frac{3}{4}$ litres of concentrated squash. In each glass a party host wants just $\frac{1}{8}$ litre of the concentrate before topping the glass up with water. The problem is the host needs to find out how may glasses of squash she can make from one bottle of the concentrate. Can you help?"

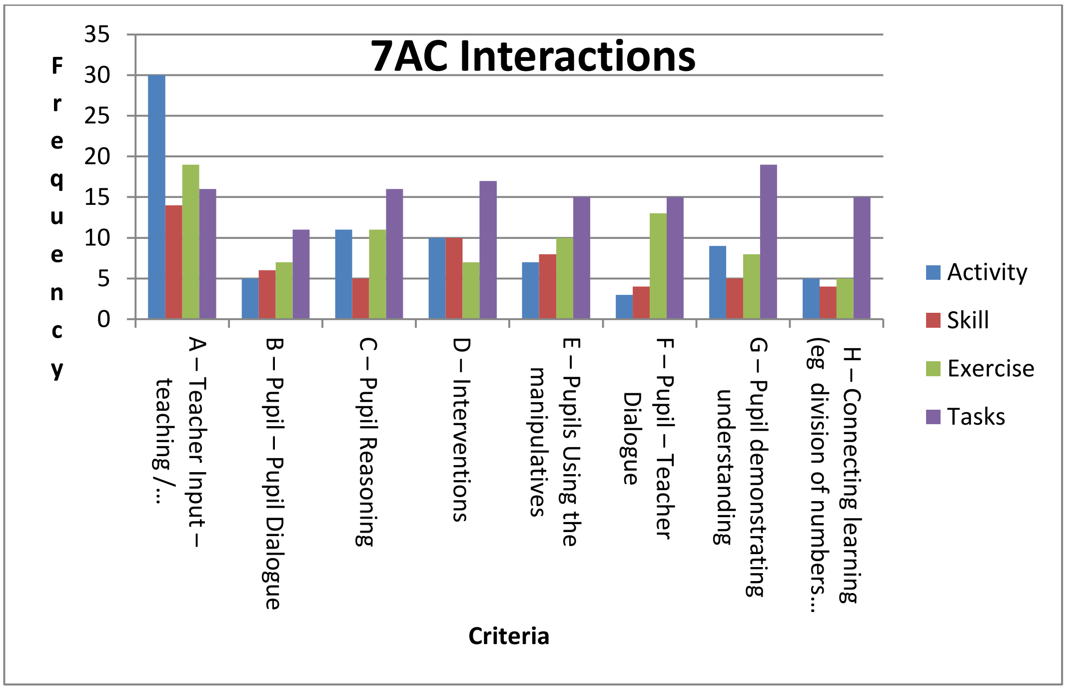


Chart 5.2.6b summarises all interactions for 7AC

It might be concluded that class 7AC's conceptual understanding of the processes of division of fractions (as developed by the skills and exercise lesson episodes) were already well formed, however they still felt the need to use the concrete manipulatives to aid their understanding of the practical context of the task. Pupils from this class gave a range of opinions relating to the manipulative tiles helping them to visualise the problem and to the relationships between fractions.

> Pupil a: They made me understand how many smaller fractions go in the bigger one.

> Pupil b: They helped because I had a visual and I could see what to do when doing the last question (the task).

Pupil c: The manipulatives] showed me how many different fractions go into a whole which helped when doing the questions

Pupil d: The manipulatives] made me understand how many lower fractions go in the bigger fraction

The visualisation of the size and relationships between fractions seems to be at the centre of the comments made by all pupils in the two classes and is independent of attainment as commented on by Shaw

> Manipulatives enhance the abilities of students at all levels to reason and communicate. Working with manipulatives deepens understanding of concepts and relationships, makes skills practice meaningful, and leads to retention and application of information in new problem-solving situations (2002, p. 3).

The findings indicate that the level of explanation given by the teacher was needed for pupils to understand the language of the question in the task and was the main reason for the interventions. The levels of literacy for the classes had been consulted and in the main their average reading age was 9 years and 7 months. The phrases “party host” and “topping up” contained in the task were unfamiliar to the pupils and these needed unravelling before they were able to access the mathematics in the task. This was not expected as the teachers had seen and ‘approved’ the text contained in the task.

Most of the teacher interventions (criteria A) in the 7AC group were to prompt and support pupils to design a more complex task for themselves. The pupils in the 7NR group were much more engaged in pupil-pupil dialogue (criteria B) during the skills part of the lessons and they were more reliant on the use of the manipulatives to support their discussions. Pupils in 7AC were more willing to demonstrate to each other (criteria G) their understanding of the partitioning of a fraction, which had been developed earlier in the lesson, during the task episode. They were able to access the mathematics in the task quickly and, as expected, the learning from the skills and exercise phases of the lesson were clearly being used to get the solution to the task.

In conclusion the findings tend to point to three outcomes.

1. Contextual tasks facilitate a more frequent and wider range of learning types, as defined by the criteria (A to H) than do activities, exercises or skills. This appears to be independent of the attainment of the pupils.

2. The use of manipulatives to solve contextual tasks is less frequently used when pupils have already achieved a level of understanding or facility with the mathematical concept.

3. There is no significant difference in pupil outcomes between high or low attaining groups when manipulatives are used either to aid conceptual understanding (developed in exercises or skills) or problem solving (developed in contextual tasks).

5.3 Summary of the Findings

This chapter has presented the findings for research question 2

(RQ2) What are the implications of a change of approach to teaching fractions for teacher training?

Research question 2 relates to the teaching of mathematics and a change in teaching approach. I have therefore also grouped these findings from this part of the study in a similar way to that in chapter 4. I have created two broad categories (and continued the numbering from the findings part 1 chapter):

3 Teaching mathematics
4 Professional development of mathematics teachers.

5.3.1Teaching mathematics,

Finding 5: From the study lessons mathematical skill development (consisting of a short teacher-led input followed by a limited number of questions with an open invitation to pupils to create and solve their own questions), gives pupils the freedom and confidence to explore and experiment with the presented skill. They then go on to explore the skill in more depth and detail than what was presented by the teacher. However, the teachers in this study rarely used this approach, nor did they use text books or worksheet problems designed in this way. Instead they took a broadly didactic approach characterised by teacher input followed by repetitive drills with no application.

Finding 6: When teachers adopted an ‘experimental’ lesson design using a sequence of learning episodes such as activity, skill, exercise and task, they did so for all age groups with the exception of those groups studying for public examinations (ages 14-16). Lesson plans revealed an absence of ‘activities’ and ‘tasks’ for this age group, with interview data from teachers revealing the reluctance to ‘spend time’ on these. They favoured the answering of past questions and the repetition of text book questions as a means to examination success. This suggests that despite the evidence that the sequence does benefit student understanding, there is still the pressure to teach to the test.

5.3.2 Professional development of mathematics teachers.

Finding 7: A trainee teacher’s degree qualification influences the way they see themselves as mathematics teachers and how they design mathematics lesson. Those teachers who studied a mathematics degree see themselves as a subject expert (a mathematician) adopting a mainly didactic teaching approach. They construct lesson plans that are teacher-led where they impart mathematical content followed by the pupils practising questions often on their own. Contrastingly those teachers who did a non-mathematics degree see themselves less as subject experts and construct lesson plans that are pupil – centred. The mathematical content is experienced and developed through social interactions with pupils often being allowed to work in groups and talk about the mathematics. Newly qualified teachers’ beliefs relating to teaching and designing mathematics lessons are not always related to, or influenced by, their own educational school experiences. Their beliefs are nevertheless influenced by the structure and ethos of the public examinations that they studied. Those teachers who did public examinations that included a coursework element were more likely to be in the group of teachers that allowed pupils to experiment and develop their own mathematical understanding.

Finding 8 : During their teacher training year the practising of skills is emphasised to trainee teachers as only one method of learning mathematics and that alternative approaches can, and are, often more successful. This study suggests that trainee teachers’ views relating to the practising of mathematics skills (either formed prior to their training year or as part of the training) are mediated by members of the department but only once they become a substantive member of the team. Mathematics subject leaders and mentors have a mediating effect on newly qualified teacher views concerning lesson design.

The corporate departmental approach is seen as the model, but newly qualified teachers often initially hold on to their original beliefs.

The next chapter will consider the implications for each of these findings from the two chapters.

Chapter 6 – Discussion

6.1 Introduction

In this chapter possible implications that stem from the findings in the previous two chapters are considered. The findings will not be individually discussed but again they will be grouped into the following:

1 Learning mathematics (RQ1)
2 Lesson design terminology (RQ1)
3 Teaching mathematics (RQ2)
4 Professional development of mathematics teachers (RQ2).

The discussion of the findings needs to be firmly placed in the context and narrative of the study. The intentions for the study were many, but the research questions arose out of the desire to use my past experiences, knowledge and understandings. I wanted to be able to inform others of how the use of precise terminology, when designing mathematics lessons, can influence pupil learning, teaching and consequently the professional development needs of teachers. Therefore this chapter brings together my thoughts which are firmly founded in the findings, with references to the literature, before offering some recommendations and thoughts in the final chapter.

6.2 Discussions relating to the Learning of Mathematics

Vygotsky's social constructivism is the underpinning learning theory on which the four learning episodes (activity, skill, exercise and task) are based in this study. In each of the 4 parts of the research lesson pupils were encouraged to work together to construct their own understandings of the division of fractions. The requirement to generate and answer their own questions gave not only ownership of the knowledge but also a shared conception and understanding of what is acknowledged to be a conceptually difficult mathematical topic. Pupils working collaboratively on well-bounded tasks, in a community of practice (Lave and Wenger, 1991), is a motivator for learning that "stems from participation in culturally valued collaborative practices in which something useful is produced"

(Illeris, 2009, p. 62). Lave and Wenger's (1991) belief that learning is "anchored in the access to participation in communities of practice with the purpose of becoming competent practitioners" (Illeris, 2009, p. 87) is at the centre of the lesson used in this study to develop pupil understanding of the division of fractions.

This study has shown that lesson structure (finding 3), together with the use of supportive resources (finding 2), expressed in the form of episodic lesson planning documents consisting of the four learning episodes (activity, skill, exercise and task) (finding 1) all have an impact on pupil learning. In particular finding 2 demonstrates that pupil learning is supported and enhanced by the inclusion of physical resources (manipulatives) as part of lessons. Research has shown that those pupils who use physical aids outperform other pupils (Greabell, 1978; Driscoll, 1983; Suydam, 1986; Raphael and Wahlstrom, 1989; Sowell, 1989; Thompson, 2012). However there is also evidence (Ball, 1992; Meira, 1998) that manipulatives do not assure conceptual understanding (Baroody, 1989; Sarama and Clements, 2016). This research demonstrates the positive effects of the use of manipulatives on learning, across the attainment range, and the support they offer pupils when posing, writing and discussing questions.

Incorporating manipulatives into lesson design can be a real challenge for teachers as there is a paucity of available physical resources targeted at pupils of secondary school age. Mendiburo and Hasselbring (2011, p. 5) reported that teachers "rarely use physical manipulatives because they are practically and pedagogically difficult to implement in classrooms". All too often the learning of mathematics, at secondary school level and beyond, is perceived and experienced by learners as a solitary, cognitive experience (Walker, 2016). Tangible physical objects, in the form of manipulatives, can help to ground abstract mathematical concepts so that they are accessible to all learners (Booth et al., 2017). Manipulatives can become almost surrogate partners in the learning experience and the importance of their use in learning mathematics should not be underestimated by teachers (Furner and Worrell, 2017).

Where departmental managers think carefully about how they resource learning with practical, physical equipment as an essential aspect of lessons, there seems to be an impact on pupil learning (Silver, 2017). Often abstract mathematical concepts are not reinforced with appropriate equipment. Expense is often cited as a restricting factor however, this need not be the case, especially when departmental managers provide teachers with time to facilitate the development and production of appropriate learning equipment. (Lau et al., 2018). Little mathematical equipment (at secondary level in particular) is available in educational suppliers' catalogues, but this research shows that creative teachers who work collaboratively, with an appropriate amount of time, do create physical resources to ground and support the learning and understanding of mathematical concepts (Mann, 2006; Rezat et al., 2018).

The contribution to research by practising teachers when using physical resources (manipulatives) to teach conceptually challenging mathematical ideas is of paramount importance. The development of a set of manipulative tiles representing all the common fractions (as indicated in this research – appendix 8) along the lines of Cuisenaire Rods, but based on a unit rectangular area, might be a useful addition to the physical resources available to teachers to aid pupils when learning about mathematical operations on fractions.

6.3 Discussions relating to Lesson Design Terminology

The three part lesson plan (starter – main – plenary), having become the de-facto approach advocated by the National Strategies early in 1999 -2000, has recently become less influential as indicated in a speech by the Ofsted Chief Inspector (Wilshaw, 2012)

> We, and in that word "we" I include Ofsted, should be wary of trying to prescribe a particular style of teaching, whether it be a three part lesson; an insistence that there should be a balance between teacher led activities and independent learning, or that the lesson should start with aims and objectives with a plenary at the end and so on and so forth [no page].

Ofsted (2012a, p. 33) finally states that "Inspectors must not expect teaching staff to teach in any specific way or follow a prescribed methodology". The move towards a three part structured lesson was seen by most teachers, at the time, as a milestone in lesson design and was heavily promoted and embraced by most teachers. Current newly qualified teachers, having experienced this type of structured lesson have started to move towards an episodic lesson design (finding 4) which allows them more freedoms. However, with freedoms inevitably comes implications, and there is now a need for teachers to be aware, that they require a deeper understanding of the characteristics of types of learning episodes that can be used when designing lessons. Given these freedoms the findings from this research suggest that when pupils experience variety in mathematics lesson structure, design and learning experiences, then the consequences are an increase in pupil motivation and learning for all attainments (finding 2).

Teachers need to be cognisant and acutely aware of the often subtle distinctions between the phrases and words that they use to describe aspects of their practice in order to effectively share common conceptions, especially when these phrases are often used interchangeably (finding 3). Teachers and subject leaders are aware that lessons with a variety of learning episodes are more likely to engage pupils but they are likely to be less aware, as this research seems to suggest, that there might be a sequence to the learning episodes for abstract mathematical concepts, such as the division of fractions, that allow optimum learning across the attainment range..

Lesson planning documents are often generic in design to allow creativity and individuality of approach across all subject departments within an institution. Bage et al. (1999) found that a uniform system of lesson planning often results in teachers not using the full range of their expertise when planning lessons. Allowing for creativity is seen as a positive of a generic lesson planning document, nevertheless, the documents should be accompanied by specific subject guidance especially in relation to appropriate learning sequences or episodes. Wilshaw (2012, no page) when speaking about lesson planning and the requirements from teachers reminds us that

> In my experience a formulaic approach pushed out by a school or rigidly prescribed in an inspection evaluation schedule traps too many teachers into a stultifying and stifling mould which doesn't demand that they use their imagination, initiative and common sense. Too much direction is as bad as too little.

Lesson plans often give teachers reminders as to lesson management issues such as health and safety issues and are scarcely ever about approaches to learning. Assessment of and for learning and subject specific vocabulary are the norm on lesson plans but this tends to be the limit of guidance given when assisting teachers in their lesson design. These features are often only included so as to be compliant with the latest learning fashion. John (2006, p. 495) reminds us that lesson plans "should not be viewed as a blueprint for action, but should also be a record of interaction". This research (finding 4) does suggest that a lesson structure consisting of the learning episodes (activity, skill, exercise and a task), in that order, where each part is firmly rooted in an understanding of the pedagogical terminology (appendix 35) does have a positive effect on pupil learning.

The role of subject leaders, Heads of Mathematics (HoM), in influencing their colleagues approaches to planning and designing lessons does not seem to have prompted a great deal of research. However, Weissglass (1991) and Milford (1998) noted that HoMs do play significant roles in facilitating change in their teachers, particularly in terms of their classroom behaviours. Maaß (2018) argues that HoMs are instrumental in designing and leading professional development for their teams and should use "lesson plans written by teachers" (p. 12) to inform practice and prompt discussions about planning and designing of the curriculum. This research indicates that a change in the design and style of a lesson, which has its foundation in a clear understanding of the terminology used in each individual learning sequence, is important in raising achievements for pupils of all attainments. As John (2006, p. 487) argues, a generic planning approach is often uninformative about the activity and "does not say enough about the uniqueness of teaching and learning". The expertise and knowledge of subject leaders in bringing about a change in lesson design through the clarification and distinctions in the use of terminology for lesson design is important and is an area for future further research.. I openly acknowledge that not all mathematics could, or even should be, taught using the learning sequence activity, skill, exercise and task as this would just perpetuate the lack of variety in mathematics lessons (John, 2006; Wilshaw, 2012). However, lessons designed using the four learning episodes (activity, skill, exercise and task) with a clear understanding of the terminology does appear to be an important finding from this research for teachers and pupils as this learning sequence does seem to have a positive impact on pupil learning across the attainment range.

6.4 Discussion relating to the Teaching of Mathematics

Subject leader beliefs (finding 8) and the enactment of their leadership role are instrumental in determining the mathematical experiences that effective departmental teachers present to pupils (Harris, 1999). Whilst subject leaders, in the main, have a corporate, collegiate approach to departmental working they are much more than this as Harris (2013, p. 6) reminds us

> The most typical management approach within an effective department is that of the 'leading professional'. This is where the head of department is considered by other departmental members as a model to follow. In short, he or she is viewed as an expert practitioner and is viewed by members of the department as a source of good practice.

In particular the newly trained teachers in this research viewed their subject leaders as the “fountain of all knowledge” and as such these newly qualified teachers would frequently align their teaching views with those of their subject leader even when they previously held very differing points of views. Weissglass (1991) and Milford (1998) have noted that subject leaders do play significant roles in facilitating change in their teachers, particularly in terms of their classroom teaching behaviours. Milford (1998) suggests that these roles involve modelling teaching, affirmation and support of the faculty members teaching.

There is little research that exists relating to the use and effects of professional language when conversing with colleagues. However, Meyer (2013) used a case study methodology to explain the role of language and collaboration in enhancement and transformation of practice, and argues that “teachers lack opportunities for talking about these changes, do not yet have a language for speaking about these changes” (p. 6). My findings (finding 6) shows that the lack of a shared pedagogical language for the teaching of mathematics among teachers and subject leaders, can lead to confusion and misunderstanding of specialist terms such as activities, skills, exercises and tasks. This can lead to teachers talking about very different pedagogical concepts, even though they appear to be able to comprehend each other’s meaning and intentions. This fuzzy use of professional language can result in differing enactments of those pedagogical meanings. This can, and frequently does, result in very different learning experiences for pupils. Marzano (2013), an independent educational consultant, argues that a common shared language should be developed for governments, schools, headteachers and teachers so as to be able to converse efficiently about the effective teaching. Schools, and school leaders, are becoming increasingly aware of the need for the development of a common language to be introduced at all levels so as to positively and effectively influence classroom teaching and professional learning (Vieira and Auriemma, 2015).

The design of a mathematics scheme of work, and sometimes even individual lesson learning sequences for pupils, is usually led and developed by subject leaders (Bengo, 2016). The use of a well-defined pedagogical language can result in lesson designs and plans that are well understood by all members of the department. An obvious implication for pupils of this commonality of use of language by teachers is the promotion of a seamless continuity in learning when they move from one teacher to another. A subject leaders' knowledge of the subject content, the teaching methods – pedagogical content knowledge, and the teaching approaches expressed through a commonly understood language is of fundamental importance when designing a rich mathematics curriculum (Enderson, Grant and Liu, 2018). Therefore, their knowledge of the subtle distinctions between learning experiences defined in terms of activities, exercises, skills and tasks is of vital importance when leading their team in lesson development. A subject leader's expertise, knowledge and use of pedagogical language is of real importance when guiding and supporting trainee and newly qualified teachers with the development of lessons (Vanblaere and Devos, 2018).

Wenger (1998) suggests that communities of practice often rehearse their collective sense of purpose by having a shared repertoire of language and practices. Furthermore Wenger, McDermott and Snyder (2002) consider that through common activities and collective learning teachers come to hold similar beliefs and values. It would seem then, that an important aspect of a subject leader's role is the use of a common, well defined, professional language to help describe their intentions and corporate meanings for other colleagues. Ghamraw (2010) suggests that subject leaders succeed by

> focusing on crafting cultures within their departments which builds a sense of common purpose, generates energy, and in which relationships are respectful and trusting. Through these relationships, subject leaders can foster teamwork, create a community and develop a collective responsibility for the learning of all students (p. 318).

The language of teaching and lesson design (finding 6) therefore needs to be carefully explored by teachers to gain a common, shared understanding of terms such as activity, skill, exercise and task. Clarity of understanding of terminology related to learning experiences is of vital importance so that when these are jointly planned by teachers there are no misunderstandings. The fuzzy use and understanding of professional language with all its connotations can lead to very different perceptions of lesson intent in the minds of teachers and ultimately very different mathematical experiences for pupils.

The findings from this research seem to suggest that learning experiences and intentions are perhaps a better methodology in which to frame lessons. These learning experiences and intentions are coded and framed in the subtle distinctions in meanings between activity, skill, exercise and task (finding 5). As such this research adds to the discussion about the role and importance of a clearly defined professional language but in a fairly specific area relating to the language used by teachers when designing lessons. This would appear to be a new area of development and research. Other researchers have mainly focused on the generic aspects of a common shared vocabulary or language but have not investigated, or tried to tightly and explicitly define, individual phrases or features of a teachers' professional language (Hauk, Jackson and Tsay, 2017).

As shown in chapter 4 section 4.4.1, teachers were clearly able to verbalise their learning intentions for pupils once they were introduced to and understand a common vocabulary. This common vocabulary is of vital importance when teachers are collaboratively planning and delivering lessons to enable a common shared understanding and consistent approach or framework for learning to take place.

Nevertheless, whilst teachers in this study found that the use of activities and tasks do enhance pupils' understanding and the application of mathematics, they were reluctant to use this approach with examination groups (finding 6). Lessons based on exercises and the practising of mathematical skills was seen as the status quo for achieving success in the public examinations. The system of school accountability in the form of public examination outcomes places huge pressures on teachers, pupils and school leaders (Perryman et al., 2011; Torrance, 2017). Whilst the significance of examinations is recognised, it should also be acknowledged that these often close down the learning and teaching opportunities in favour of a narrow, one dimensional approach to the learning of the examination content. Policy, rhetoric and support documentation relating to public examinations is often described and valued in the grade descriptor outcomes to be achieved by pupils. Little attention or advice is given to teachers as to the methods for teaching content or lesson designs to support teaching and learning. The result of this lack of advice, in the form of sharing good practice, is often seen in the quality and variety of learning experiences and intentions offered to pupils in the examination years. There is a lack of will on the part of teachers to share “their knowledge and experiences with one another, and this negatively affects learner performance” (Arends, Winnaar, and Mosimege, 2017, p. 8) and this has an impact on learners.

A teacher having the confidence to follow their professional beliefs and convictions when designing learning experiences for pupils, irrespective of the system which measures outcomes, is perceived to be difficult or perilous given the current accountability systems. Pupils often experience boredom, stress and pressure in their final years when studying for examinations. The pleasure and joy of learning often disappears as a consequence of the restricted learning approaches used by teachers in order to cover the examination content. Boaler (2008) demonstrated that innovative approaches to the teaching of mathematics can be reliably used with the possibility of still achieving good examination results. This research also demonstrates the influence that a change in teaching design can have on pupil motivation, engagement and outcomes.

An activity, at the start of a lesson, designed around pupils in which they are required to recall prior knowledge, based firmly in a social constructivism learning paradigm, would appear to motivate and engage pupils of all attainments. Initially pupils find an activity, where “no assistance / instructions" are given by the teacher, to be a difficult concept. This is mainly because of the freedoms, or lack of bounds, that this type of open activity requires. However, a change towards a pupil centred mathematics lesson rather than lessons directed or conducted by the teacher, does appear to provide variety and this research would also seem to suggest that this approach increases pupil motivation.

Similar results to those in this study relating to pupils posing and answering their own questions and activities based on the recall of prior knowledge were found by others (Christiansen and Walther, 1986; Mason, 1991; Chapman, 1997). However little has changed in the intervening years, as this research also suggests. Teachers are more concerned with a particular lesson structure or style rather than creating a favourable learning environment in which communication between pupils is the norm. There is an absolute need to break the mould of a prescribed lesson format. There is also a need for pupils and teachers to take control and assume a variety of lesson formats to support learning. For example, as a first step teachers need to consider a change to the normal opening or lesson starter which is often designed as a challenge or a ‘settling-down’ question. A move towards the structure used for this research might be the first step for the brave practitioner.

Confidence and self-efficacy of teachers to guide pupils to appropriate questions and the all-encompassing need to cover the mathematics syllabus, especially when studying for public examination, prevents a culture of active promotion of pupils taking responsibility for devising their own questions and then finding solutions (Coe et al., 2014). Taking the decision to allow pupils the freedom to explore requires teachers to be extremely confident both in pupils actively engaging in the learning process and also in their own subject and pedagogical expertise. However, most mathematical skills questions are teacher predefined either from a textbook or a worksheet. The implications of allowing pupils the freedom to select and design questions appears to aid their self-confidence and efficacy of mathematics. Allowing total pupil autonomy may not be desirable as misconceptions are easily developed and become difficult to eradicate once learnt. The findings here allow pupils to explore and develop their mathematical understanding but in a controlled manner and a fairly tight and self-contained mathematical topic. When pupils do pose questions in fairly self-contained topics research has found that pupil confidence, understanding and motivation are increased and it is a useful approach “for teachers to assess students' cognitive processes, identify misconceptions and modify instruction” (Irvine, 2017, p. 387).

The ability to independently pose real-world problems needs to be promoted in both institutional and national policies. However, the first step might be for policy makers to signal very clearly to the profession that wider aspects of pupil learning such as the pupils' ability to write their own question on a prescribed topic are as important as the demonstration of correctly worked solutions to pseudo real-life questions. Indeed, it is a key theme in the literature that pupils "should have opportunities to formulate problems and questions that stem from their own interests" NCTM (1989, p. 67), however my research and experience demonstrates that this rarely happens in practice.

6.5 Discussion relating to the professional development of mathematics teachers.

This section considers findings 7 and 8 relating to teacher perceptions based on their academic qualifications and the mediating effects subject leaders have on staff. A teacher's belief about what they can teach is often perceived through the lens of their own academic qualifications (mainly at degree level); this has been witnessed in a number of studies (Fennema and Franke, 1992; Thompson, 1992; Ma, 1999, Karatas et al., 2017). Current national policy is to recruit highly qualified teachers with degrees in the subject they wish to teach. Additionally a number of alternate routes allow those without a subject specific degree to enter the profession after attending an intense subject knowledge course. These courses are often only a few weeks in length and therefore cannot and do not give participants the same body of knowledge as those who enter the profession with subject degrees. However, it was clear from these findings that the impact of a mathematics degree has on the self and professional image of the teacher and the supplementary effect this has in their teaching practice (Ernest, 1995).

Very competent teachers in this study, who did not have a mathematics degree, were often reluctant to engage in the teaching of mathematics at higher levels. Furthermore the findings suggested that a teacher's academic profile influences the design of their lessons which in turn impacts on pupil learning. Ernest (1989) argues that a “teacher's displayed attitudes to the teaching of mathematics, such as enthusiasm and confidence, can be expected be a major contributor to the ethos of the mathematics classroom. This in turn can be expected to have a powerful influence on pupils' perceptions of mathematics” (p. 26).

It is important for subject managers and leaders to realise that

> teachers differ greatly in their effectiveness, but teachers with and without different qualifications differ only a little. Therefore, according to this school of thought, policies that emphasize motivating principals and increasing their discretion over hiring should replace policies that require particular qualifications (Wayne and Youngs, 2003, p.108).

My analysis of the teacher interview data in this study shows that teachers with a degree in mathematics believe their pupils can and will achieve a confidence in the subject if they teach rules and algorithms. Alternatively those teachers without a degree in mathematics believe their pupils will only be able to achieve if they fully understand the concept irrespective of the teaching approach. However, my findings also show that, despite their lack of self confidence in their own level of mathematical attainment, there was no evidence that pupil's learning is negatively affected. Indeed there is some evidence that these teachers more than compensate for their relatively lower mathematical knowledge base by designing lessons in which pupils feel they are equal partners in the learning and that the teacher is mathematically proficient. What is perhaps more important than a mathematics degree is the teacher's deep understanding of lesson structure and learning episodes. Yet this is in direct contrast to the Teacher Education and Development Study in Mathematics findings (Tatto, et al., 2012) which calls for mathematics teachers to have a university mathematics degree as a requirement. Indeed, they point out that this has been one of the main goals of teacher education policy in many countries over the years and this has "thus affected teacher recruitment and the subsequent experience of these teachers once they are employed" (p. 206). Interestingly in the UK in 2018 there has been a relaxation in the need for a subject specific degree, but this is due to a teacher shortage in some subjects rather than any empirically based research.

So these findings (7 and 8) from the study have implications for recruitment policy and for the management of mathematics departments in relation to staffing. Policy makers and managers need to value, and make explicit, the need for pedagogical knowledge alongside, and equal to, subject knowledge when recruiting new staff. This would not only have a positive effect on how highly competent non mathematics graduates see themselves, it would also raise the aspirations of department heads to consider the balance between colleagues with and without mathematics degrees when recruiting teachers. When making selection decisions departmental managers also need to consider carefully whether a mathematics degree leads to the best teacher. The emphasis on a high level of subject knowledge (Carter, 2015) as a requirement for teaching is important, but the current definition of subject knowledge equating to academic qualifications is not always helpful.

The influence of our public examination system on the design of mathematics lessons extends far beyond the subject content. Teachers' beliefs about good mathematical lesson design are often side-lined in favour of a strategy designed to 'cover' and 'deliver' the contents of the mathematics examination. Pedagogical beliefs have become well established by the time students begin their teacher training. There is evidence that their views begin to develop during their time as pupils where they decide who is, and is not, an effective teacher (Applefield, Huber and Moallem, 2000). Teachers' beliefs have a direct impact on pupils' learning as evidence in my research findings and others (Stitt-Gohdes, 2001). Teachers often revert to a teaching style and approach influenced by the examination system they experienced at school. Nevertheless, belief systems are dynamic and subject to change (Muijs and Reynolds, 2015) so if we want to change pedagogical beliefs and classroom practices, it might be necessary to change the assessment system (Webb and Cox, 2004).

Bold curriculum innovations with social constructivist philosophies such as those embodied in the School Mathematics Project (SMP) and Midland Mathematics Experiment (MME) have given way to the ever increasing need for pupils to gain certain public examination grades with a more didactic learning philosophy. This finding would suggest that it is of paramount importance to policy makers, and in particular those who set the public examination philosophy, that their influences on the beliefs of the next generation of teachers are encoded carefully into the design of the public examinations. The findings seem to suggest that the examination system has a multi-faceted role and is far wider than just the outcome measure of pupils' ability to learn examination specification content.

Managers should not underestimate their influence over departmental colleagues. In this study newly qualified teachers were often eager to impress departmental managers to gain professional recognition and promotion. They appeared to acquiesce (or tacitly agree) and abandon their own beliefs where these were at odds with the managers. Keay (2009) found that "school subject departments provide the setting for influential professional development and that experienced teachers strongly influence their newly qualified colleagues" (p. 225). Monsour (2000) observed that supportive leaders significantly influence the experiences of teachers in their first year of teaching. Departmental subject leaders therefore have a hugely difficult role in shaping the learning for pupils whilst supporting and developing expertise in their colleagues. The inclusion of ideas, beliefs and pedagogical knowledge of colleagues, and in particular those newly qualified teachers who often bring with them the latest research, is fundamental to the development of the profession and in particular the next generation of practitioners (i.e. the pupils).

Johnson, Peters and Williams (1999) reported on research where university academics, who had no prior connections to the teachers, were working collaboratively alongside teachers to enhance professional development. The research reported differing expectations of the academics and school teachers and this resulted in tensions. The collaborative, trusted coaching partnership at the heart of this research was considered by all those involved to be a significant reason for the success we achieved. Initial Teacher Education Tutors who have a year-long connection with newly qualified teachers are perhaps ideally situated to perform this supportive professional development. With continued access to the newly qualified teachers, when visiting the next cohort of trainee teachers, they are ideally situated to facilitate and encourage classroom research.

Alternative approaches for supporting newly qualified teachers have been proposed, for example in France newly qualified teachers "should benefit from at least five weeks of training at an IUFM during their first two years in service, in order to underpin their first steps in the career" (Cros and Obin, 2003, p. 40). In 2002, the United Kingdom introduced a 3 year pilot project "Early Professional Development Scheme" for teachers in the second and third years of teaching, which followed the induction year, with a remit of strengthening teacher autonomy through school support, mentor and the Local Education Authority support. At the time the local education authority mentors were subject experts and "helped raise the profile and extend understanding of the multitude of development activities available, through communications with schools and teachers directly" (Moor et al., 2005, p. viii).

As a final thought, Ernest (2002, p. 2) argues that empowerment in the learning of mathematics is achieved by “implementing long-term programmes through which learners develop the mathematical capabilities, the skills of using and applying mathematics, and confidence and a sense of personal ownership of mathematics”. It is my belief that the findings from this research do demonstrate Ernest’s (2002) view of pupil empowerment when they are learning mathematics.

Chapter 7 – Final Thoughts

This chapter provides a brief summary of the study and suggests possible questions and plans for future research studies. The summary reflects on the research study involving teachers and evaluates how I conducted the research. I also reflect on how I changed because of this research.

7.1 Brief Summary of the Study

The aim of this study was to investigate the influences of lesson design on pupil learning and the implications for teachers, especially in relation to the precise use of pedagogical terminology by mathematics teachers. The study investigated the effects of the use of pedagogical terminology whilst designing a lesson to teach the division of fractions to pupils across the attainment range in a single school setting. The initial research model was developed based on a background questionnaire data from 201 trainee teachers and 21 qualified teacher mentors. A pilot lesson was videoed, prior to the research, with a small group of pupils in the study school, together with their teacher who later participated in the full study.

As part of the full study five participant teachers (table 4.2.2) were interviewed using the same semi-structured interview protocol. At the end of each of the two study lessons pupils were asked to give their views on the separate learning episodes of the lesson that they were engaged in (tables 4.6, 4.6a and 4.6b). Lesson plans were examined from a number of the participant teachers to compare how they were using pedagogical terminology prior to and after the study. The study lessons, using pupils with widely differing attainment levels, were videoed using multiple cameras and then analysed (appendices 33 and 34) to assess the level of mathematical understanding given the clear usage of the pedagogical terms under investigation.

Based on the outcomes from this study, which relates to the learning of the division of fractions across the attainment range with 11-12 year olds, lessons do seem to be positively influenced when they are constructed using a clear understanding of the differences between the four pedagogical terms activity, skill, exercise and task. The use of a manipulative, in the form of a fraction tile, by pupils whilst engaged with all four pedagogical terms, was not initially considered to be a variable or a line of enquiry for this study. However, in the pupil reflections collected at the end of the lesson, and from the analysis of the video footage, they were seen to be a positive factor in engaging and supporting learners.

The results of the study also indicate that teachers participating in research with researchers can be used to help teachers develop deeper understandings of pedagogy as well as how pupils learn. This participative research engagement, by teachers, is particularly important as a means of professional development and is indicative of the current professional climate. It is hoped that the study will further the understanding of pedagogical terminology and how this can positively affect pupil learning, in this case with a conceptually difficult mathematical topic such as the division of fractions.

As a consequence of me sharing my lesson designs and the lesson video footage with the teachers involved in the study, it became apparent that the terminology that I used in the design and delivery of the lessons was not being interpreted in the same way by teachers. There were differences in understanding and deploying the terms activity, skill, exercise and task. Therefore, the focus of the study did move from investigating the teaching of fractions into an exploration of the significance of pedagogical terminology, and more importantly the importance of clear shared meanings when designing learning as expressed in lesson plans. Without clarity and precision of meaning, it is almost impossible to design truly effective, collaborative, shared lesson plans and to engage in truly meaningful, fundamental discussions around lesson pedagogy.

7.1.1 Limitations

The study school, the whole mathematics department together with the senior leadership of the school, and the pupils were well known to me from the school improvement work I had undertaken with them over a number of years prior to the study as a Local Authority Education Mathematics Adviser. Nine of the eleven departmental teachers (table 3.3.1) had been trained through the initial teacher training courses I lead and hence it might be considered that they would be loyal and give responses I might be expecting. However this was not the case. They did participate in the study as critical professionals and voiced their views and opinions in an open and frank manner. This I considered to be a real strength. Whilst I was reflecting on why we did reach a shared understanding of the terminology under investigation, I came to the conclusion that relationships founded on professional trust and authority were of significant importance during the open, frank and critically reflective discussions.

The selection of the school and the participating teachers was considered a positive as the trust levels between them and me were high; however it can also be seen as a negative if viewed from the perspective of a power relationship between lecturer and ex-trainee teachers. Harrison, Dymoke and Pell (2006) remind us that this power relationship exists between beginning teachers and their induction tutors as “there is an element of power dependency because one has more knowledge and experience than the other” (p. 1055). In fact the relationship between lecturer and trainee is no different to that between researcher and teacher. There definitely was not any power relationship between lecturer and the ex-trainee teachers, as participants were all free to give their opinions and actively encouraged to participate by giving their views in an atmosphere of open professional development. The participants voiced their opinions, openly and frankly so as to challenge each other and me in order to gain a deeper collective understanding of the 4 pedagogical terms. Participant research relies on anti-authoritarian relationships where new knowledge is created from collective understandings and the notion of the researcher having “superior” knowledge is challenged (Karnieli-Miller, Strier and Pessach, 2009).

The study lessons were designed with the content and approach being ultimately determined by me. There was minimal input from the participant teachers, they did however supply the learning approaches that the pupils had experienced and were familiar with. The only other items they were asked for, and they willingly supplied, were the pertinent academic and social details of the pupils that were to be involved in the study. As the sole designer of the lessons there was no ambiguity in the use of the terminology. My interpretation of these terms was clear and consistent with those formulated from the literature and defined at the end of chapter 2. This consistency was based on many years as a lead practitioner. Also from previous roles as a demonstrator of lessons, I had learnt the importance of clarity of meaning, the precision of use of the terminology, for both the teachers observing practice and the pupils learning. I considered that being the sole designer of the lessons to be both simultaneously a strength and a limitation. A strength because I was able to focus clearly on the aspects of lesson design terminology under investigation, but a limitation as I did not fully involve the other professionals at this stage of the research.

The two study classes, at different ends of the attainment range, were randomly selected by the participating teachers. When their performance data was analysed prior to selection there was nothing unusual about the groups and they were considered by all involved to be representative of the whole cohort. With significantly more time a larger number of classes from across the attainment and age range would have been involved. However, a larger scale project was beyond the capacity of both me and the participating teachers. Undertaking the classroom actions was not the limitation but analysing a wider range of pupil data and resulting video evidence would have taken significantly more time. Even though this time investment would have given a richer set of results it was considered by all involved to be of only limited use for the considerable time investment needed.

The selection of just one school setting, two classes and a small number of teachers might be viewed as a significant limitation of this research. Williams (2007, p. 70) reminds us that the justification for this approach is that a “researcher purports to provide in-depth insight into a phenomenon, [then] the researcher might view selecting a small but informative sample, which is typical of qualitative research”.

Despite these limitations, the results of the study do have positive implications for future research into pedagogical lesson terminology. This study provides a foundation for larger research studies examining the efficacy of defining pedagogical lesson terminology.

7.1.2 Plans for future research

Having thought about, and reflected on, the outcomes of the study I began to start to consider the future. It seemed to me that pupils would gain a greater self-efficacy and fondness for mathematics if the culture and pedagogical language used when designing lessons could be embedded into teacher training programmes. If the quality of lesson designs could be improved and made more consistent through the precise use of pedagogical terminology, then it is my opinion that pupil engagement and learning of mathematics would be improved.

It is fairly easy to change one's own practice and it requires only your own efforts. To bring about wider changes in practice requires both professionals within your own institution (which again is relatively easy to achieve) and for the wider profession (which is more difficult to affect) to collaborate and comprehend shared meanings. Changing teacher perceptions about lesson design requires considerably more effort and research, but it is achievable as Inouye (2017) found when working with a group of science teachers.

The findings of this study have generated some insights that suggested to me future avenues for research. A logical next step might be that a subsequent cycle of research should be carried out to understand how a change in a mathematics teacher's perceptions, and use of precise pedagogical terminology, influences the outcomes of pupils measured by formal national examinations. In view of my findings from this study three further questions arise that would warrant action:

1. How does a change in lesson design using precise pedagogical terminology affect pupil outcomes as measured in public examinations?

2. How do we encourage a change in the teaching of mathematics topics, which have fairly unanimous single teaching approaches, and encourage alternative teaching methods based on clear use of pedagogical terminology?

3. How do we encourage mathematics teachers to be brave and trial, adopt and utilise alternative, non-standard, teaching approaches which are based on clear definitions of pedagogical lesson terminology?

In order to address the first of these three questions we would need to look at one single mathematics topic, use an alternative approach from the norm, and observe and analyse the effects on pupils of all attainments in their public examinations. The selected topic would need to be common across the two tiers of the public examination system (for example a topic such as simple trigonometry, or Pythagoras' Theorem). The selection of a common topic which spans the two tiers of the public examination system would ensure that all attainments would be included in the research. Additionally, these two topics are frequently taught by most mathematics teachers, using fairly consistent and standardised approaches. Hence, these topics might be open to a change in lesson design. The resulting outcomes could then be compared against those from the more standardised approaches.

In order to discover answers to the second question we would need to investigate topics that have a "prescribed" or generally accepted way of being taught, and then encourage teachers to adopt an alternative approach. There are a number of topics especially in number and algebra that would lend themselves to this sort of investigation (for example, solving simultaneous equations; long multiplication of number or solving linear equations).

The final question is much more difficult, it is about legitimacy and self-belief which are neither easy to measure nor change. The environment in which a teacher works needs to be supportive and open to experimental change. Such supportive environments are not always easy to achieve given the pressures exerted by the current public examination system. Additionally it is possibly it will be even more difficult for core a subject, such as mathematics, on which the reputation of the school is often measured in league tables. Nevertheless, this is a poor reason for us not to be trying to find an answer to this question.

7.2 Reflections on the Research Project

In this sub section I return to the questions that I still feel need further exploration, having reviewed the mathematics education literature, to see if my research study had provided any answers. I then reflect briefly on issues that I had not expected at the outset but which the research revealed.

As this study is grounded in a qualitative case study research methodology, with participating teachers, it is predicated on the critical engagement with change which naturally incorporates planning and implementation in a real world situation. The evaluation of the outcomes viewed through the lens of an iterative cycle of problem diagnosis; planning for action and implementation is a way of finding out about a complex system whilst trying to change it (Elden and Chisholm, 1993). Education is receptive to this methodological approach (Greenwood and Levin, 2006; Reason and Bradbury, 2008; McNiff, and Whitehead, 2011) and this makes the approach particularly useful when engaging in research with other professionals. The participating teachers and I agreed that this study did bring about change in their practice, and as such I was pleased that it was not merely an academic exercise. We did agree that the influence on their practice benefited pupils and their learning. In my opinion the improvement in learning should be at the heart of research when it involves children, or in fact any learner.

We (the participating teachers and I) also agreed that at the outset of the study we lacked a common shared understanding of some of the terminology. This study does identify a lack of an existing body of theory relating to pedagogical terminology. As the study progressed we did agree that pedagogical terminology might be a significant weakness when conversing and sharing lesson plans. I feel that this study has made a contribution to starting to fill that gap in constructing a clearer understanding of the four pedagogical terms (activity, skill, exercise and task) when used in mathematical education. Early work on defining mathematical aspects of tasks have been made and reported on by Swan (2005); Mason and Johnston-Wilder (2006); Watson and Mason (2007); Ainley (2008), but clarity of terminology and more work and research is needed to further define the finer distinctions between the terms. The findings from this study could be built upon by the systematic investigation into a wider set of pedagogical terminology especially when applied to lesson planning. In particular the shared understandings of the terminology by teachers when discussing their practice relating to the designing of lessons might then enhance pupil performance due to a consistency of approach from their teachers.

Whilst reflecting, during and at the end of this study, on the use of lesson design terminology such as activities, skills, exercises and tasks I came to the conclusion that teachers do require a deep understanding of their meaning. The implication of this conclusion would be the need for teachers to continuously explore the terminology during and after qualification. I firmly believe that the involvement of teachers in research does have a direct relevance to classroom practice. In fact I would go as far as to say it should be a requirement. The participating teachers in this study were suggesting that this might be part of a local or even a national strategy. Continuing professional development is, and has been, a concept that has resonance for the education professional. Given the current lack of co-ordinated national policies around professional development the opportunity to reconnect with pedagogical terminology to develop a richer, deeper, understanding does exist. Moreover, since the group of participating teachers are representative of the profession as a whole, it would appear that there is a desire and willingness to engage in such professional development activities for both their own self benefit and that of the learner.

The participating teachers expressed a need for continuing professional development to be linked to research and that a system should be established to sustain conversations and research between teachers, school managers and researchers. Those involved in this study at the end concluded that continuing professional development that explicitly acknowledges a practitioner's broadening, deepening and interconnected pedagogy needs to be at the forefront of research so as to positively influence the learning in classrooms and the outcomes for all pupils.

7.2.1 Unexpected outcomes revealed by the study.

I had expected at the outset of the research to experience some resistance to a change in moving from a purely algorithmic 'flip and invert' method of teaching the division of fractions. This method is so ingrained in the psyche of mathematics teachers, and exemplified in all textbooks, that a change in approach based on equivalence of denominators was expected to be met with a fair amount of resistance and scepticism. In actuality all participating teachers were open and receptive and looking for a different way of teaching the topic. I also expected the two mathematics classes to be a little shy or reticent to engage in the study. The preparations with the two classes prior to the research, by the participating teachers, and the fact that they were present videoing the lesson was a positive action and encouraged the pupils to enthusiastically engage with the work. Post study we agreed that this strategic approach was useful, and probably contributed positively to the pupil interactions seen in the videos.

7.3 How I have changed

Through the process of developing alternative learning materials for a commonly taught topic, that allowed all pupils access, it has given me a sense of purpose and self-belief about the teaching of mathematics. I had a long held belief that some of the ways I taught mathematical topics were more effective than the standard traditionally accepted approaches and this study confirmed my belief for one topic. The move away from an algorithmic and isolated way of dealing with the division of fractions, which is poorly understood and ill-remembered by pupils, to an approach which is more logical and connected to other topics worked, and I was able to demonstrate the effects in a research setting. Also, I informally returned to the school to visit the teachers and the pupils after nearly a year. A good number of the pupils remembered the method for the dividing of fractions that I had taught. Of course there could be multiple reasons for this, such as the further exposure to the methods after the study had concluded, but it was satisfying to note the impact of the research.

One of the most gratifying aspects of the study was working with other teachers and influencing their views on the teaching of mathematics. Additionally to observe the changes in the participating teachers through their shared conversations, and the lesson plans they developed, was an aspect I had hoped to witness. In practice their active involvement, enthusiasm and willingness to take on board new ideas and use and develop them, was amazing and refreshing. I would venture this was principally due to their willingness to be part of the research.

7.3.1 Improvement in my practice

The involvement in developing and implementing an alternative lesson design and approach to the teaching of fractions has had a favourable effect on my development. Those beliefs I held about the precise use of lesson design terminology when planning lessons for the teaching of mathematics do now have some foundation in research. The consequence is the influence on my day to day practice whilst lecturing the next generation of mathematics teachers. For this study I developed a set of fraction manipulative tiles and during the period of the research I also developed manipulative tiles for other mathematical topics. These tiles have been trialled with pupils. These tiles are now widely used for the teaching of fractions by the participating teachers and students who are training to be teachers.

In terms of my teaching skills, I believe I have become more confident in my own convictions that some of the ways in which I design, and therefore teach topics, are valid and appropriate (Nolan and Molla, 2017). Since the start of the study I have become much more open about expressing my beliefs and ideas about how to design lessons using precise terminology. The resulting effect on the teaching of mathematical topics and is that I am more critical of some of the ways that some topics are presented to learners. Having a deeply held set of belief about teaching is important. I now firmly believe it is the underpinning research that gives legitimacy, and possibly more importantly credibility, to these beliefs. This is yet another reason why, in my opinion, all practitioners should be involved with classroom research.

7.3.2 My self-perception and increased self esteem

I have already mentioned that confidence in my own pedagogical skills and knowledge has increased dramatically during this study. I now seek opportunities, both in the lecture room and whilst in schools visiting trainee teachers and their mentors, to discuss alternative strategies related to lesson designs which hopefully develop their teaching practice. Working with a group of teachers, all of whom have now gained promotion, either internally or at other schools, helped my efforts to disseminate the outcomes of this study and raise awareness of alternative approaches to the teaching and learning of mathematics. I know this study has been effective in encouraging colleagues in other schools as I have witnessed the lesson design learning episodes (activities, skills, exercises and tasks) based on this research whilst visiting trainee teachers.

The most personally rewarding part of the research, other than the pleasures of seeing pupils of all attainments succeed, is that I am now seen by colleagues in the university, and those in schools, as a mathematics teacher and learning innovator rather than a mathematician. Due to the research I feel more at ease with the role of a teacher innovator, which was quietly dormant and suppressed, than the public face of a mathematician. Whilst my love of mathematics and the inner curiosity of proving the next piece of mathematics will never go, I have found an equally rewarding way of spreading the news about the sublime beauty of the subject. This new role as a mathematics teaching and learning innovator suits my personality and my beliefs about mathematics, but much more importantly how mathematics should be presented to a learner. I have no convictions about being the "maverick" in the room when discussing alternative approaches to the designing of lessons for the learning and teaching mathematics.

7.4 Evaluation of the Research

Whilst evaluating this research I am mindful that the findings are based on a single educational setting, in just two classrooms, and with a small sample of newly qualified teachers. The broad nature of the two research questions which investigated the influences of pedagogical design terminology on practice, together with a qualitative case study research methodology, could have generated a variety of outcomes. I am therefore aware that the analysis of the data collected in this study has generated just one of a variety of possible interpretations.

Designing and planning lessons, far from being a collegiate activity, is usually a solitary endeavour; often an idiosyncratic activity; designed to be fit for purpose and is highly personal in nature. As noted by Butt (2008, p. 3) "underneath the surface of a good lesson lies the bedrock of teacher understanding about the principles of sound pedagogical practice". Yet when lesson plans are shared by teachers then phrases such as activity, skill, exercise and task written by the lesson plan originator can take on completely different manifestations, interpretations and delivery by the receiving teacher. This study did demonstrate that once a common pedagogical language is shared by teachers then lesson plans do result in a shared understanding to the benefit of the learner. As Butt (2008, p. 4) reminds us “one of the main difficulties when planning lessons is achieving a clear definition of what we, as teachers, are trying to convey to the students about our subject”.

The participating teachers in the study informally compiled a compendium of professional vocabulary relating to lesson design. This allowed the group to exemplify shared meanings of pedagogical terminology and in particular those phrases that relate to lesson design. This seemed an obvious action and would be the first step for all groups of teachers who might be considering any form of group professional development around lesson design.

7.4.1. My contribution to knowledge

This research has contributed to the current body of knowledge of mathematics teaching and learning in three distinct areas: lesson design terminology, pupil learning and teacher practice. I would therefore make three tentative claims about how this research adds to the body of knowledge of the teaching and learning of mathematics.

In the literature review I argued that the links between learning theories and key components of teaching such as lesson planning, design and pedagogical terminology have not been sufficiently developed. This research indicates a link between the precise use of pedagogical terminology and lesson planning demonstrating that pupils from across the attainment range can successfully access and learn difficult and normally poorly understood mathematical topics. The use of a social constructivist model for teaching situated in a community of practice as the basis for the study lessons is not new or even unusual, but grounding clarity of shared understandings (teacher to teacher and teacher to pupil) of specific pedagogical terminology, which are then consistently used in the delivery of lessons, is new. In this study I have started to consider and offer definitions of four pedagogical terms (activities, skills, exercises and tasks) which relate to the teaching of mathematics and I would argue these are worthy of further development and additional research.

Secondly, when supported by appropriate physical items (manipulatives) pupils are able to pose their own questions and problems and successfully answer them irrespective of their attainment. Pupils are easily able to pose questions ranging from simple practice and drill to more complex real life tasks when a lesson is structured using precise pedagogical terms. Additionally the freedom afforded by the use of manipulatives and the lesson structure does appear to offer pupils the opportunity to pose and investigate mathematics of their own choosing inside the topic being studied.

Finally the study suggests that the self-efficacy of teachers in respect of their academic qualifications does have an impact on their professional practice and further it influences the way in which they design, plan and implement lessons. Their previous academic experiences either as a pupil at school or a student on academic courses has an influence on the ways in which they view their own abilities in the subject. This self-efficacy is important as teachers frame and design their lessons based on their level of academic qualification.

7.5 Recommendations

Policy should promote and continually emphasise evidential practice and in particular the importance of the acquisition of a professional language which has clear definitions and distinctions in the meanings of words and phrases that are commonly used and misused by practitioners.

Subject leaders need to be continually updating their knowledge and skills relating to lesson design. Currently once teachers have undertaken their training then there is little or no support (in terms of time) given to them to keep themselves up to date with current thinking and research. There is some support available through the professional subject associations and the National Council for Teaching and Excellence in Mathematics (NCTEM), but the use of this support usually requires subject leaders to invest time outside of their professional hours. Evidence from the interviews with subject leaders in the study suggested that there is a need for national policy leaders and politicians to think carefully about how teachers should be supported to continually update their classroom skills and pedagogical knowledge. The subject leaders were in favour of a system similar to the medical profession whereby teachers are required to undertake an annual one week study leave so that their registration as a teacher continues. The study leave should be external to the institution as current INSET (In Service Training Days) are often perceived by their departmental teachers as not training in the true sense. This recurrent updating of skills is of paramount importance to teachers and by implication pupils. Currently, whilst qualifying, a teacher learns how to construct, design and write a lesson plan. This might be the only time in their 40 year career that they visit these skills, as expressed by the subject leaders in the research school who offered the view that policy makers cannot allow this to continue. The subject leaders and the national figure also agreed that there needs to be a national expectation that a teacher will frequently undertake professional development in relation to current research around lesson design and terminology.

Currently, in 2018, national teaching standard 4 (DfE, 2012, p.11) deals with planning and teaching well-structured lessons. Whilst the standard is somewhat helpful in trying to describe actions in relation to lesson planning and design, it does lack the detail in relation to the terminology needed to formulate plans

1. impart knowledge and develop understanding through effective use of lesson time
2. promote a love of learning and children's intellectual curiosity
3. set homework and plan other out-of-class activities to consolidate and extend the knowledge and understanding pupils have acquired
4. reflect systematically on the effectiveness of lessons and approaches to teaching
5. contribute to the design and provision of an engaging curriculum within the relevant subject area(s).

These statements about lesson planning are generic and open to subjective interpretation. Furthermore they do not provide practical help on the construction of lesson plans or the precise language needed to communicate a teacher's lesson intentions to other colleagues.

Whilst policy makers set targets (often as a crude measure of examination outcomes) they should also carefully consider other important aspects of outcomes from learning. The participant teachers suggested that these targets do not always coincide with the rationales that mathematics teachers hold when embarking on a professional life in teaching. Whether a teacher's rationale for studying mathematics comes from the polar opposites of either a purist (the beauty of the subject) or a utilitarian (usefulness to the individual and society) it is nevertheless “an inherently social activity” (Schoenfield, 1992, p. 335) where problems are posed and solutions found (or not found). Ernest (2002) holds the view that mathematically empowered learners should be mathematics creators and that learners should pose and solve problems. Pólya (1981) also stresses the need for pupils to be mathematical experimenters and that teachers can create conditions so that pupils develop creativity and work independently to find mathematical solutions. It would therefore be advantageous for policy makers to express targets using a different language which values and stimulates and values a variety of teaching approaches which might not be measured by examinations outcomes.

Teacher professional development is most likely to occur when collaborative opportunities for the team members exist so that they can learn from each other. Day (1993) considers reflection as central for teachers to improve practice rather than simply collecting a bank of knowledge. Whilst reflective praxis from all involved was at the heart of this research, four additional key characteristics are required from teachers for effective staff development as identified by Joyce and Showers (1980)

1. Presentation of theory or description of new pedagogical skills
2. Coaching for application
3. Modelling and demonstration of teaching
4. Practice in simulated classrooms

The later three were all voiced by the teachers involved in this research and hence the findings from this research highlight the same challenges and opportunities that Joyce and Showers found in 1980. The trusted and skilled partnership (Hopkins et al., 1997, p. 7) at the heart of this research resulted in “reflection about teaching” and “generated new ideas of working” in a collaborative reflective, modelling and coaching culture. Steadman et al. (1995, p. 49) argued that “change in the classroom which involves more than extending the repertoire by acquiring new skills will mean changing attitudes, beliefs and personal theories”. This research did change the attitudes and beliefs of the teachers and was influential in providing the basis for further explorations of personal and collective pedagogy by the participating teachers. All those involved in the research have now completed a master’s degree.

7.6 Summary and conclusion

My final thoughts having reviewed the study are that lesson design and in particular pedagogical language or terminology which describes lessons are difficult complex subjects and not commonly understood by the profession. Changes need to be made to teacher training programmes to include pedagogical terminology if we are to achieve an improvement in pupil and trainee teacher learning. It was naive to believe that this study would have a major impact on the status quo, but the small change, with a few teachers for a limited number of pupils, was worthwhile and the findings it produced are useful for continuing research.

Having continued to work with those teachers involved in the study, and the departments that they have eventually moved on to, is that implementing change is initially relatively easy with enthusiastic participants, but sustaining the progress is infinitely more difficult. I am convinced that is not a unique feeling or phenomenon. Having spoken to, and subsequently worked with, the pupils involved in the study the learning that did take place, and which can be recalled by these pupils, did produce some small changes in the learning of the division of fractions. The study has also produced a number of teachers who do not teach the division of fractions by using a trick or algorithm. Hopefully the findings that relate to lesson terminology have produced the initial conversations for future research by the participating teachers. It may be the way forward in defining a pedagogical language for the design of lessons by mathematics teachers to the benefit of future generations of learners.

I am convinced that a teacher participating in research to change their own practice and therefore influence the practice of their colleagues is the way forward. I hope that this small step I have made will encourage others to join in the research for the ultimate benefit of mathematics education and the learning of future generations of pupils.

References

Abbott, A. (1988). *The system of professions: An essay on the division of labor*. University of Chicago Press.

Abramovich, S. (2015). *Mathematical Problem Solving as a Link between Algorithmic thinking and conceptual knowledge*. The Teaching of Mathematics. **18**(2), pp. 45–60.

Abrar, M. (2017). A Critical Evaluation of Qualitative Reports and Their Contributions to Educational Research. *PAROLE: Journal of Linguistics and Education*, **7**(1), pp. 13-22

Ackerman, P. L. (1996). A theory of adult intellectual development: Process, personality, interests, and knowledge. *Intelligence*, **22**(2), pp. 227-257.

Adey, P. (1988). *Cognitive Acceleration: Review and prospects*, International Journal of Science Education, **10**(2), pp. 121-134.

Adhami, M., Johnson, D. C., and Shayer, M. (1997). Does 'CAME' work? summary report on the phase 2 of the cognitive acceleration in mathematics education. *Proceedings of the British Society for Research into Learning Mathematics Conference*, **15**, pp. 26-31.

Adhami, M., Johnson, D. C., and Shayer, M.(1998). Thinking Maths: the programme for Accelerated Learning in Mathematics. Oxford: Heinemann Educational Books.

AfL. (2017). *Professional learning : Examples of learning intentions.* [Accessed 24 August 2017] <Available at : http://www.assessmentforlearning.edu.au/professional_learning/learning_intentions/learning_examples_intentions.html>.

Ainley, J. (2006). Purposeful task design: an algebraic example. In *Anais do SIPEMAT. Recife, Programa de Pós-Graduação em Educação-Centro de Educação– Universidade Federal de Pernambuco*. p p. 1-12.

Ainley, J. (2008). Task Design based on Purpose and Utility. *Topic Study Group 34: Research and Development in Task Design and Analysis, The Eleventh International Congress on Mathematics Education,* Monterrey, Mexico.

Ainley, J., and Margolinas, C. (2015). Accounting for student perspectives in task design. *Task design in mathematics education.* Springer. pp. 115-141.

Ainley, J., Bills, L., and Wilson, K. (2004). Constructing meanings and utilities within algebraic tasks. In M. J. Høines and A. B. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education.* Bergen, Norway: PME, **2**, pp. 1-8.

Ainley, J., Bills, L., and Wilson, K. (2005). Purposeful task design and the emergence of transparency. In Chick, H.L. and Vincent, J. L. (Eds.), *Proceedings of the Twenty Ninth Annual Conference of the International Group for the Psychology of Mathematics Education*, *.* Bergen, Norway: PME, **2**, pp. 17-24.

Ainley, J., Pratt, D., and Hansen, A. (2006). Connecting engagement and focus in pedagogic task design. *British Educational Research Journal.* **32**(1), pp. 23–38.

AITSL (2014). *Australian Professional Standards for Teachers* [Accessed 12 May 2015] <Available at : https://www.aitsl.edu.au/teach/standards>.

Alvarez, R., and Urla, J. (2002*). "Tell Me a Good Story: Using Narrative Analysis to Examine Information Requirements Interviews during an ERP Implementation."* The DATA BASE for Advances in Information Systems, **33**(1), pp 38-52.

Amador, J. (2016). Mathematics pedagogical design capacity from planning through teaching. *Mathematics Teacher Education and Development.* **18**(1), pp. 70-86.

Anderson, J. (2006). Some teachers' beliefs and perceptions of problem solving. *Paper presented at the Proceedings of the 29th Conference of the Mathematics Education Research Group of Australasia,* Melbourne.

Anderson, R. D. (1996). *Study of Curriculum Reform. [Volume I: Findings and Conclusions.] Studies of Education Reform*. US Government Printing Office, Superintendent of Documents; Washington, DC.

Andrews, P., and Hatch, G. (2000). A comparison of Hungarian and English teachers' conceptions of mathematics and its teaching. *Educational Studies in Mathematics,* **43**(1), pp. 31-64.

Anthony, S., and Jack, S. (2009). Qualitative case study methodology in nursing research: An integrative review. *Journal of advanced nursing*, **65**(6), pp. 1171-1181.

Applefield, J.M., Huber, R., and Moallem, M. (2000*).* Constructivism in theory and practice: toward a better understanding. *High School Journal,* **84**(2), pp. 35–59.

Archer, M. (2012). *The reflexive imperative in late modernity*. Cambridge, UK: Cambridge University Press.

Arends, F., Winnaar, L., and Mosimege, M. (2017). Teacher Classroom Practices and Mathematics Performance in South African Schools: A Reflection on TIMSS 2011. *South African Journal of Education*, **37**(3), pp. 1–11.

Argyris, C., and Schön, D.A. (1991). Participatory action research and action science compared: A commentary. In W. F. Whyte (Ed.) *Participatory action research*. California: Sage Publications, Inc. pp. 85- 96.

Ari, B.A., and Enosh, G. (2012). Power relations and reciprocity: dialectics of knowledge construction. *Qualitative Health Research*, **23**, pp. 422–429.

Arksey, H., and Knight, P. T. (1999). *Interviewing for social scientists: An introductory resource with examples.* Sage.

Armstrong, A., Ming, K., and Helf, S. (2018). Content area literacy in the mathematics classroom. *The Clearing House: A Journal of Educational Strategies, Issues and Ideas*, *91*(2), 85-95.

Arriaza, P., Nedjat-Haiem, F., Lee, H. Y., and Martin, S. S. (2015). Guidelines for Conducting Rigorous Health Care Psychosocial Cross-Cultural/Language Qualitative Research. *Social Work in Public Health,* **30**(1),pp 75–87.

Artigue, M., and Perrin-Glorian, M. (1991). Didactic engineering, research and development tool: Some theoretical problems linked to this duality. *For the Learning of Mathematics,* **11**(1), pp. 13-18.

Ashton, P. T., Webb, R. B., and Doda, N. (1983). *A study of teachers' sense of efficacy*. University of Florida, Gainesville.

Aslam, M., and Kingdon, G. (2013). How teachers' pedagogic practice influences learner achievements : A study from the Punjab Pakistan. In B. Moon (Ed.), *Teacher education and the challenge of development: A global analysis.* Routledge pp. 164-182.

Atkins, C., and Sampson, J. (2002). Critical appraisal guidelines for single case study research. ECIS 2002 Proceedings, **15**, pp. 100-109.

Aubrey, K. and Riley, A. (2018). *Understanding and using educational theories*. SAGE Publications Limited.

Ausubel, D. P. (1968). *Educational psychology: A cognitive view*. New York: Holt, Rinehart and Winston New York.

Avenier, M. J., and Thomas, C. (2015). Finding one's way around various methods and guidelines for doing rigorous qualitative research: A comparison of four epistemological frameworks. *Systèmes d'Information et Management (French Journal of Management Information Systems),* **20**(1).

Avis, J., Orr, K. and Tummons, J. (2010). Theorizing the work-based learning of teachers: In Avis, J., Fisher, R. and Thompson, R. (Eds.), *Teaching in lifelong learning: A guide to theory and practice.* Maidenhead: Open University Press.

Bage, G., Grosvenor, I., and Williams, M. (1999*).* Curriculum planning: prediction or response? A case study of teacher planning conducted through partnership action research. *Curriculum Journal,* **10**(1), pp. 49–70.

Bakker, A., and Smit, J. (2017). *Theory development in design-based research: An example about scaffolding mathematical language.* Making change happen, Springer. pp. 111-126.

Baliram, N. and Ellis, A.K. (2019). The impact of metacognitive practice and teacher feedback on academic achievement in mathematics. *School Science and Mathematics*.

Ball, D. L. (1992*).* Teaching mathematics for understanding: What do teachers need to know about the subject matter? In Kennedy, M. (Ed.), *Teaching academic subjects to diverse learners*. New York: Teachers College Press. pp. 63–83.

Ball, D. L., and Wilson, S. M. (1990). Knowing the subject and learning to teach it: Examining assumptions about becoming a mathematics teacher. *Paper presented at the Annual Meeting of the American Educational Research Association*, Boston.

Ball, D. L., McDiarmid, G., Houston, W., and Sikula, J. (1990). *Handbook of research on teacher education.* The Subject-Matter Preparation of Teachers. New York, NY: Macmillan,

Ball, D. L., Thames, M. H., and Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, **59**, pp. 389–407.

Bandura, A. (1982). Self-efficacy mechanism in human agency. *American Psychologist*, **37**(2), pp. 122-147.

Barbosa, J.C., and Pereira de Oliveira, A.M. (2013*).* Conflicts in designing tasks at collaborative groups. In Margolinas, C. (Ed.). (2013). *Task Design in Mathematics Education*. Proceedings of ICMI Study 22. Oxford. pp. 543-550.

Barnes, A., Venkatakrishnan, H., and Brown, M. (2003). *Strategy or strait-jacket? teachers' views on the English and mathematics strands of the key stage 3 national strategy*. London. Association of Teachers and Lecturers.

Barnes, M., Clarke, D., and Stephens, M. (2000). Assessment: the engine of systematiccurriculum reform. *Journal of Curriculum Studies,* **32**(2), pp. 623 – 600.

Baroody, A. J. (1989). *‘Manipulatives don’t come with guarantees’. Arithmetic Teacher*, **37**(2), pp. 4–5.

Baroody, A. J. (2003). The development of adaptive expertise and flexibility: The integration of conceptual and procedural knowledge. In Baroody, A. J. and Dowker, A. (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise*. London: Erlbaum. pp. 1-33.

Bartlett, F. C. (1932). *Remembering : a study in experimental and social psychology.* Cambridge: Cambridge University Press.

Basic Skill Agency. (1997). *International numeracy survey: A comparison of the basic skills of adults 16-60 in seven countries*. London: Basic Skill Agency.

Basit, T.N. (2010). *Conducting research in educational contexts.* Bloomsbury Publishing.

Bassey, M. (1999). *Case study research in educational settings*. McGraw-Hill Education (UK).

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., Tsai, Y. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress*. American Educational Research Journal*, **47**(1), pp. 133-180.

Baumfield, V. Hall, E., and Wall, K. (2008). *Action research in the Classroom.* Sage Publications. London.

Beard, C. M., and Wilson, J. P. (2006). *Experiential learning: A best practice handbook for educators and trainers*. Kogan Page Publishers.

Becker, J. P., and Shimada, S. (Eds.) (1997). *The open-ended approach: A new proposal for teaching mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Behr, M. J., Harel G., Post T., Lesh R. (1993*). Rational numbers: towards a semantic analysis-emphasis on the operator construct.* In Carpenter T.P., Fennema E., Romberg T.A. (Eds.) (1993). *Rational numbers: as integration of research.* Hilsdale (N.J.): Lawrence Erlbaum. pp. 13-47.

Behr, M. J., Lesh, R., Post, T., and Silver, E. (1983*),* 'Rational number concepts'. In R. Lesh and M. Landau (Eds.), *Acquisition of Mathematics Concepts and Processes*. Academic Press, New York, pp. 91–125.

Behr, M. J., Post T., Silver E., Mierkiewicz D. (1980). *Theoretical foundations for instructional research on rational numbers.* Proceedings PME 1 pp. 60-67.

Behr, M. J., Post T., Wachsmuth I. (1986). Estimation and children's concept of rational number size. In: Shoen H.L., Zweng M.J. (Eds.) (1986). *Estimation and mental computation.* 1986 Yearbook. Reston (Ma): NCTM.

Behr, M.J., Wachsmuth I., Post T. (1985). Construct a sum: a measure of children's understanding of fraction size. *Journal for research in mathematics education*, **16**, p p. 120-131.

Beijaard, D., Verloop, N., and Vermunt, J. D. (2000). Teachers' perceptions of professional identity: An exploratory study from a personal knowledge perspective. *Teaching and Teacher Education*, **16**, pp. 749–764.

Bell, S., Brown, P., and Buckley, S. (1989). *New York cop and other investigations: A resource book for mathematics teachers*. London: Cambridge University Press.

Bengo, P. (2016). Secondary mathematics coaching: The components of effective mathematics coaching and implications. *Teaching and Teacher Education,* **60**, pp. 88- 96.

Bengtsson, M. (2016). How to plan and perform a qualitative study using content analysis. *NursingPlus Open*, **2**, pp. 8–14.

Bennett, N., Newton, W. Wise, C., Woods, A., and Anastasia, E.(2003). *The Role and Purpose of Middle Leaders in Schools*. NCTL. [Accessed 1 May 2016] <Available at : http://dera.ioe.ac.uk/5118/13/download_id%3D17365%26filename%3Drole-purpose-of-middle-leaders-in-schools-summary_Redacted.pdf>.

Berg, B. L. (2001). *Qualitative research methods for the social sciences. Boston*. Allyn and Bacon .

Berg, B. L., and Lune, H. (2012). *Qualitative research methods for the social sciences (8th Ed.)*. Pearson Education.

Bergman, D.J. (2018). Prospective Teachers' Perceptions of Influential Teacher Qualities. *The Advocate*, **23**(6), p. 6.

Biggs, J. (1999). *Teaching for quality learning at university.* (Buckingham, society for research into higher education and open university press). Teaching for Quality Learning at University.

Bishop, A. J. (1991). *Mathematical enculturation: A cultural perspective on mathematics.* Kluwer Academic Publishers. The Netherlands.

Blaxter, L., Hughes, C., and Tight, M. (2002). *How to research*. Open University Press, Buckingham: UK.

Bloom, B. S. (1959). *Taxonomy of Educational objectives,* Handbook New York David. Domain, Mckay Group Inc.

Boaler, J. (2002). *Experiencing school mathematics: Traditional and reform approaches to teaching and their impact on student learning.* Routledge.

Boaler, J. (2008). Promoting 'relational equity' and high mathematics achievement through an innovative mixed-ability approach. *British Educational Research Journal*, **34**(2)**,** pp. 167–194.

Boaler, J. (2009). *The elephant in the classroom: Helping children learn and love maths.* London: Souvenir Press.

Bokhove, C., and Jones, K. (2014). Mathematics textbook use in England: Mining Ofsted reports for views on textbooks. In K. Jones, C. Bokhove, G. Howson, and L. Fan (Eds*.), Proceedings of the International Conference on Mathematics Textbook Research and Development (ICMT-2014.)* Southampton: University of Southampton, pp. 159–166.

Booth, J. L., McGinn, K. M., Barbieri, C., Begolli, K. N., Chang, B., Miller-Cotto, D., Young, L. K., and Davenport, J. L. (2017). Evidence for cognitive science principles that impact learning in mathematics. In D. C. Geary, D. B. Berch, R. Ochsendorf, and K. M. Koepke (Eds.), *Acquisition of complex arithmetic* *skills and higher-order mathematics concepts* . Cambridge, MA: Academic Press, pp. 295–325.

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., and Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education*, pp. 194-222.

Borko, H., Peressini, D., Romagnano, L., Knuth, E., Willis-Yorker, C., Wooley, C., Masarik, K. (2000). Teacher education does matter: A situative view of learning to teach secondary mathematics*. Educational Psychologist,* **35**(3), pp.193-206.

Borovik, A. V., and Gardiner, T. (2007). *Mathematical abilities and mathematical skills.* Manchester: The University of Manchester, MIMS.

Borwein, J.M. (2002). The Experimental Mathematician: The Pleasure of Discovery and the Role of Proof. In Simmt, E. and Davisd, A. (Eds.), *Proceedings of the ACTES Annual meeting.* Queen’s university. Ontario Canada.

Boston, M. D., and Smith, M. S. (2009). Transforming secondary mathematics teaching: Increasing the cognitive demands of instructional tasks used in teachers' classrooms. *Journal for Research in Mathematics Education*, pp. 119-156.

Bouck, E.C., and Park, J. (2018). A Systematic Review of the Literature on Mathematics Manipulatives to Support Students with Disabilities. *Education and treatment of Children*, **41**(1), pp. 65–106.

Boulet, G. (1998). ‘Didactical implications of children’s difficulties in learning the fraction concept’. *Focus on Learning Problems in Mathematics,* **20**(4), pp. 19–34.

Bowe, J., and Gore, J. (2017). Reassembling teacher professional development: The case for quality teaching rounds*. Teachers and Teaching,* **23**(3), pp. 352-366.

Bowl, M., Cooke, S., and Hockings, C. (2008). Researching across boundaries and borders: the challenges for research. *Educational Action Research*, **16**(1), pp. 85-95.

Boyd, B. A. (1992). *The relationship between mathematics subject matter knowledge and instruction: A case study.* Department of Mathematical Sciences, San Diego State University.

Boydell, T. H. (1976). *Experiential learning: Manchester monograph number 5.* Manchester, UK: Department of Adult Education, University of Manchester.

Breen, M. P. (1987). Learner contributions to task design. In C. N. Candlin, and D. F. Murphy (Eds.), *Language learning tasks.* Englewood Cliffs: Prentice Hall. pp. 23-46.

Britton, S. (2002*).* Are students able to transfer mathematical knowledge. *Paper presented at the Proceedings of the 2nd International Conference on the Teaching of Mathematics. New York: Wiley.*

Brooks, J., and Brooks, M. (1993). *The case for the constructivist classrooms.* Alexandria, VA: ASCD.

Broome, R. (1991). Wissenstypen und professionelles Selbstverstandniss [Types of knowledge and professional self-concept]. *Zeitschrift fur Padagogik*, **37**, pp. 769-785.

Brown, A. (2002). Conceptualizing Limit in Calculus (Why is it still a problem?). In Simmt, E., and Davisd, A. (Eds.), *Proceedings of the ACTES Annual meeting.* Queen's university. Ontario, Canada.

Brown, J. S., Collins, A., and Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, **18**(1), pp. 32-42.

Brown, M. W. (2009). Toward a theory of curriculum design and use: Understanding the teacher-tool relationship. *Mathematics Teachers at Work: Connecting Curriculum Materials and Classroom Instruction,* pp. 17-36.

Brown, M., Brown, P., and Bibby, T. (2008*).* "I Would Rather Die": Reasons Given by 16-Year-Olds for Not Continuing Their Study of Mathematics. *Research in Mathematics Education,* **10**, pp. 3-18.

Brumbaugh, D. K., and Rock, D. (2013). *Teaching secondary mathematics (3rd Ed.)*. Mahwah, New Jersey: Lawrence Erlbaum Associates.

Bruner, J. S. (1964). The course of cognitive growth. *American Psychologist,* **19**(1), pp.1-15.

Bruner, J. S. (1996). *The culture of education*. Harvard University Press.

Bryman, A. (2008). *Social Research Methods.* Oxford University Press. Oxford.Yates

Buerk, D. (1990). Writing in mathematics: A vehicle for development and empowerment. *MAA Notes*, **16**, pp. 78-84.

Burghes, D. (2011). *International comparative study in mathematics training:* Recommendations for initial teacher training in England. Reading: CfBT Education Trust.

Burnard, P. (1991). A method of analysing interview transcripts in qualitative research. *Nurse Education Today*, **11**, pp. 461–466.

Burns, M., and Silbey, R. (2000*). So you have to teach math? Sound advice for K-6 teachers.* Sausalito, CA: Math Solutions Publications.

Burton, D. and Bartlett, S. (2005). *Practitioner Research for Teachers*. London: PCP

Burton, L. (1984). *Thinking things through: Problem solving in mathematics.* Oxford, UK: Basil Blackwell.

Burton, L. (1989). Images of mathematics. *Mathematics Teaching: The State of the Art*, pp. 180-187.

Burton, L. (1998). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics,* **37**(2), pp. 121-143.

Burton, L. (2005). *Methodology and methods in mathematics education research: where is the "why"?* In S. Goodchild and L. D. English (Eds.), *Researching mathematics classrooms: a critical examination of methodology.* Greenwich, CO: Information Age Pub. pp. 1-10.

Burton, L. (2012). *Learning mathematics: From hierarchies to networks* Routledge.

Burton, R. M., and Obel, B. (2011). Computational modeling for what-is, what-might-be, and what-should-be studies – and triangulation. *Organization Science*, **22**, pp. 1195-1202.

Butt, G. (2008). *Lesson planning 3rd edition*. Bloomsbury Publishing.

Buxton, L. (1981*). Do you panic about mathematics?* London, Heinernann.

Cai, D., Viljaranta, J., and Georgiou, G. K. (2018). Direct and indirect effects of self-concept of ability on math skills. *Learning and Individual Differences*, **61**, pp. 51-78.

Calderhead, J., and Robson, M. (1991*).* Images of teaching: Student teachers' early conceptions of classroom practice. *Teaching and Teacher Education,* **7**(1), pp. 1-8.

Campbell, A., and Groundwater-Smith, S. (Eds.) (2007). *An ethical approach to practitioner enquiry: dealing with issues and dilemmas in action research.* Abingdon: Routledge.

Capel, S. A., Leask, M., and Turner, T. (2001). *Learning to teach in the secondary school: A companion to school experience.* Taylor and Francis.

Capraro, M. M., Bicer, A., Grant, M. R., and Lincoln, Y. S. (2017). Using precision in STEM language: A qualitative look. *International Journal of Education in Mathematics,Science, and Technology*, **5**(1), pp. 29-39.

Carducci, R., Pasque, P. A., Kuntz, A. M., and Contreras-McGavin, M. (2013). Disrupting Façades of Clarity in the Teaching and Learning of Qualitative Research. *Qualitative Research in Education,* ***2***(1), pp.1 -26.

Carnine, D., Jitendra, A., and Silbert, J. (1997). A descriptive analysis of mathematics curricular materials from a pedagogical perspective. *Remedial and Special Education,* **18**, pp. 66-81.

Carolyn, A. M. (1991). Is dealing with mathematics as a thoughtful, subject compatible with maintaining satisfactory test scores? A nine-year study. *Journal of Mathematical Behavior*, **10**, pp. 215-248.

Carpenter, T. P., and Moser, J. M. (1982). The development of addition and subtraction problem-solving skills. *Addition and Subtraction: A Cognitive Perspective*, pp. 9-24.

Carr, D. (2000). *Professionalism and ethics in teaching*. London: Routledge.

Carter, A. (2015). *Carter Review of Initial Teacher Training (ITT)*. HMSO. London.

Carter, B., and Sealey, A. (2009). Reflexivity, realism and the process of casing. *The Sage handbook of case-based methods*, pp. 69-83.

Cerbin, W., and Kopp, B. (2006). Lesson study as a model for building pedagogical knowledge and improving teaching. *International Journal of Teaching and Learning in Higher Education,* **18**(3), pp. 250-257.

Chapman, A. (1993). Language and learning in school mathematics: A social semiotic perspective*. Issues in Educational Research*, **3**(1), pp. 35-46.

Chapman, O. (1997). Metaphors in the teaching of mathematical problem solving. *Educational Studies in Mathematics,* **32**(3), pp. 201-228.

Charalambous, C. Y., and Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understandings of fractions. *Educational studies in mathematics*, **64**(3), pp. 293-316

Charalambous, C. Y., Hill, H. C., and Mitchell, R. N. (2012). Two negatives don't always make a positive: Exploring how limitations in teacher knowledge and the curriculum contribute to instructional quality. *Journal of Curriculum Studies,* **44**(4), pp. 489-513.

Chevallard, Y. (1999*).* L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches En Didactique Des Mathématiques,* **19**(2), pp. 221-265.

Chikiwa, C., and Schäfer, M. (2018). Promoting Critical Thinking in Multilingual Mathematics Classes through Questioning. *EURASIA Journal of Mathematics, Science and Technology Education*, **14**, p. 8.

Chin, C., and Kayalvizhi, G. (2010). Posing Problems for Open Investigations: What questions do pupils ask? *Research in Science and Technological Education,* **20**(2), pp. 269-287.

Christiansen, B., and Walther, G. (1986). Task and activity. In B. Christiansen, A. G. Howson, and M. Otte (Eds.), *Perspectives on mathematics Education*. Dordrecht, Netherlands: Reidel. pp. 243-307.

Clandinin, D. J., and Rosiek, J. (2007). Mapping a landscape of narrative inquiry. In D. J. Clandinin (Ed.), *Handbook of narrative inquiry: Mapping a methodology,* pp. 35–75. Thousand Oaks: Sage Publications.

Clark, L. M., DePiper, J. N., Frank, T. J., Nishio, M., Campbell, P. F., Smith, T. M., and Choi, Y. (2014). Teacher characteristics associated with mathematics teachers' beliefs and awareness of their students' mathematical dispositions. *Journal for Research in Mathematics Education,* **45**(2), pp. 246-284.

Clarke, D. (2006). Fractions as division: The forgotten notion? *Australian Primary Mathematics Classroom,* **11**(3), pp. 4-10.

Clarke, D. M., Roche, A., and Mitchell, A. (2008). Ten practical tips for making fractions come alive and make sense. *Mathematics Teaching in the Middle School,* **13**(7), pp. 372-380.

Clarke, J. (2011). Changes in the ‘beliefs’ of pre-ITE maths students on a 24-week subject knowledge enhancement course. *Research in Secondary Teacher Education,* **1**(1), pp. 3-8.

Clements, D. H. (2000). From exercises and tasks to problems and projects: Unique contributions of computers to innovative mathematics education. *The Journal of Mathematical Behavior,* **19**(1), pp. 9-47.

Clements, M. A. (1984). Language factors in School Mathematics. In P. Costello, K. Ferguson, M. Stephens, D. Trembath and D. Williams (Eds.), *Facets of Australian mathematics education.* Adelaide: Australian Association of Mathematics Teachers. pp. 137-148.

Clont, J. G. (1992). *The concept of reliability as it pertains to data from qualitative studies.* Paper Presented at the annual meeting of the South West Educational Research Association. Houston, TX.

Cobb, P. (1986). Contexts, goals, beliefs and learning mathematics. *For the Learning of Mathematics,* **6**(2), pp. 2-9.

Cobb, P., and Bauersfeld, H. (1995). *The Emergence of Mathematical Meaning: Interactions in Classroom Cultures.* Lawrence Erlbaum Associates, Hillside, NJ.

Cobb, P., and Bowers, J. (1999). Cognitive and situated learning perspectives in theory and practice. *Educational Researcher*, **28**(2), pp. 4-15.

Cobb, P., and Jackson, K. (2011). Towards an empirically grounded theory of action for improving the quality of mathematics teaching at scale. *Mathematics Teacher Education and Development*, **13**(1), pp. 6-33.

Cockcroft, W. H. (1982). *Mathematics Counts.* HM Stationery Office London.

Coe, R., Aloisi, C., Higgins, S., and Major, L. E. (2014*). What makes great teaching? Review of the underpinning research.* London, UK: Sutton Trust. [Accessed 30 April 2017] <Available at: http://www.suttontrust.com/wp-content/uploads/2014/10/What-Makes-GreatTeaching-REPORT.pdf >.

Cohen, L., Manion, L., and Morrison, K. (2013). *Research methods in Education*. Routledge.

Coles, A., and Brown, L. (2013*).* Making distinctions in task design and student activity. *Paper presented at the Task Design in Mathematics Education. Proceedings of ICMI Study 22,* Oxford, pp.184-195.

Coles, A., and Brown, L. (2016). Task design for ways of working: Making distinctions in teaching and learning mathematics. *Journal of Mathematics Teacher Education*, **19**(2), pp. 149-168.

Collie, R. J., Martin, A. J., Bobis, J., Way, J., and Anderson, J. (2018). How students switch on and switch off in mathematics: exploring patterns and predictors of (dis) engagement across middle school and high school. *Educational Psychology*, pp. 1-21.

Collins, A. (1996). Design issues for learning environments. *International Perspectives on the Design of Technology-Supported Learning Environments,* pp. 347-361.

Cooney, T. (2002). Mathematics teacher change and developments. In Leder, C. G., Pehkonen, E. and Torner, G. (Eds.), *Beliefs: A hidden variable in mathematics education?* Springer . pp. 127-147.

Corbin, J., and Strauss, A. L. (2008). *Basics of qualitative research (3rd ed.).* Thousand Oaks.

Cornall, J. (1986). The small school: Achievements and problems. *Education Today,* **36**(1), pp. 25-36.

Cornelius, M., and Ernest, P. (1991). The philosophy of mathematics Education. *British Journal of Educational Studies,* **39**(3), p. 348.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., and Vidakovic, D. (1996). Understanding the limit concept: beginning with a coordinated process schema*. Journal of Mathematical Behavior*, **15**, pp. 167–192.

Craig, H. J., Kraft, R. J., and Du Plessis, J. (1998). T*eacher development: Making an impact .* Washington DC, Agency for International Development.

Creswell, J. W. (2007). *Qualitative inquiry and research design: Choosing among five designs.* Thousand Oaks, CA: Sage.

Creswell, J. W. (2009). *Research design: Qualitative, quantitative, and mixed methods approaches.* Thousand Oaks, CA: Sage publications.

Cros, F., and Obin, J. P. (2003). *Attracting, developing and retaining effective teachers country background report for France*. [Accessed 21 March 2015] <Available at :http:// www.oecd.org/france/30414627.pdf>.

Cross, D. I. (2009). Alignment, cohesion, and change: Examining Teachers' beliefs Their influence on instructional Practice. *Journal of Mathematical Practice,* **12**, pp. 325-346.

Cutcliffe, J. R., and Ramcharan, P. (2002). Levelling the playing field? exploring the merits of the ethics-as-process approach for judging qualitative research proposals. *Qualitative Health Research,* **12**(7), pp. 1000-1010.

Darling-Hammond, L. (2000). Teacher quality and student achievement. *Education Policy Analysis Archives*, **8**(1), pp.1-44.

Darling-Hammond, L., and Bransford, J. (Eds.) (2005). *Preparing teachers for a changing world: What teachers should learn and be able to do.* San Francisco. Jossey-Bass.

Darlington, Y., and Scott, D. (2003). Qualitative research in practice: Stories from the field. *Social Work Education: The International Journal*, **22**(1), pp. 115-118.

Darragh, L. (2016). Identity research in mathematics education. *Educational Studies in Mathematics*, **93(**1), pp.19-33.

Davis, B., and Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. *Journal for Research in Mathematics Education,* **34,** pp.137–167.

Davis, E. A., and Krajcik, J. S. (2005). Designing educative curriculum materials to promote teacher learning. *Educational Researcher,* **34**(3), pp. 2–14.

Davis, G., Hunting, R. P., and Pearn, C. (1993), 'What might a fraction mean to a child and how would a teacher know?' *The Journal of Mathematical Behaviour,* **12**(1), pp. 63 – 76.

Davis, R. B. (1984). *Learning Mathematics: The Cognitive Science Approach to Mathematics Education.* Norwood, NJ:

Davis, R. B., Maher, C. A., Noddings, N. (1990). *Suggestions for the improvement of mathematics education.* In Davis, R.B., Maher, C.A. and Noddings, N. (Eds.), *Constructivist views on the teaching of mathematics journal for research in mathematics education monograph no. 4.* Reston, VA: National Council of Teachers of Mathematics. pp. 187-191.

Dawe, L. (1983). Bilingualism and mathematical reasoning in English as a second language. *Educational Studies in Mathematics,* **14**(4)*, pp. 325-353.*

Day, C. (1993). Reflection: A necessary but not sufficient condition for professional development. *British Educational Research Journal,* **19**(1), pp. 83-93.

Dearing, R. (1994). *The national curriculum and its Assessment School.* London . Curriculum and Assessment Authority. HMSO.

Dembélé, M., and Miaro-II, B. (2003). *Pedagogical renewal and teacher development in sub-Saharan Africa: A thematic synthesis*. Paper presented at the ADEA Biennial Meeting, pp. 3-6.

Denzin, N. K. (1973). *The research act: A theoretical introduction to sociological methods.* Transaction publishers.

Denzin, N. K. (1989b). *Interpretive interactionism*. Newbury Park, CA: Sage.

Denzin, N. K., and Lincoln, Y. S. (2011). *Strategies of qualitative inquiry* (Paperback Edition). London: Sage.

Department for Education (DfE) (2013). *National Curriculum in Mathematics programmes of study: key stage 1 to 4: National curriculum in England.*

Department for Education and Employment (DfEE) (2001*). Key Stage 3 National Strategy: Framework for Teaching Mathematics, Years 7, 8 and 9.* HMSO. London. . [Accessed 26 July 2016] <Available at: https://www.sussex.ac.uk/webteam/gateway/file.php?name=teachers-standards-department-for-education.pdfandsite=26>.

Department of Education and Science (DES) (1988). *Education Reform Act.* London: HMSO.

Derry, S. J. (2007). Guidelines for video research in education: Recommendations from an expert panel. *Chicago, IL: University of Chicago, Data Research and Development Center*.

Desimone, L. M., Smith, T., Baker, D., and Ueno, K. (2005*).* Assessing barriers to the reform of US mathematics instruction from an international perspective*. American Educational Research Journal*, **42**(3), pp. 501-535.

DfBIS. (2016). *Impact of poor English and maths skills on employers : Literature review. (No. 267).* London: Department for Business, Innovation and Skills.

DfE (2012). Teachers' Standards. [Accessed March 2015]< Available at : https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/301 107/Teachers__Standards.pdf>.

DfE(2014). *Statutory Guidance. National Curriculum in England: Framework for Key Stages 1 to 4.* [Accessed 12 May 2015] <Available at :

https://www.gov.uk/government/publications/nationalcurriculum-in- england-framework-for-key-stages-1-to-4/the-nationalcurriculum-in-england-framework-for-key-stages-1-to-4#the-nationalcurriculum-in-england>.

Dickens, J. (2016). *NAO teacher recruitment report: The 15 key points.* [Accessed 26 July 2016] <Available at : http://schoolsweek.co.uk/nao-teacher-recruitment-report-the-15-key-points/>.

Dickerson, V. C., and Zimmerman, J. L. (1996), *Myths, misconceptions, and a word or two about politics (Special edition on narrative,* J. L. Zimmerman and V. C. Dickerson, (Eds.), Journal of Systemic Therapies.

Dinham, S. (2013*).* The quality teaching movement in Australia encounters difficult terrain: A personal perspective. *Australian Journal of Education,* **57**(2), pp. 91-106.

Doan, R. E. (1997). Narrative therapy, postmodernism, social constructionism, and constructivism: Discussion and distinctions. *Transactional Analysis Journal*, ***27***(2), pp. 128-133.

Donmoyer, R. (2012). Can qualitative researchers answer policymakers' what-works question? *Qualitative Inquiry*, **18**(8), pp. 662-673.

Dörfler, W. (2006). *Inscriptions as objects of mathematical activities. In New mathematics education research and practice*, (ed.) A. Maas, and W. Schloeglmann. Rotterdam/Taipei: Sense Publishers. pp. 97-111.

Dougiamas, M. (1998). A journey into constructivism. [Accessed 15 September, 2015]<Available at: https://dougiamas.com/archives/a-journey-into-constructivism>.

Doyle, D. (2007). *Transdisciplinary engagement: researching with rather than on.* In: Campbell, A. and Groundwater-Smith, S. (Eds.) (2007). An ethical approach to practitioner enquiry: dealing with issues and dilemmas in action research. Abingdon: Routledge, pp. 75–87.

Doyle, W. (1983). Academic work*. Review of Educational Research,* **53**(2), pp. 159-199.

Drever, E. (1995*). Using semi-structured interview in small-scale research-A teacher's guide.* UK: The Scottish Council for Research in Education.

Dreyfus, H. L., and Dreyfus, S. E. (2004). The ethical implications of the Five-Stage Skill-Acquisition Model. *Bulletin of Science, Technology and Society*, **24**, pp. 251–264.

Driscoll, M. J. (1983). *Research within reach: Elementary school mathematics and reading.* St. Louis: CEMREL.

Driver, R., Asoko, H., Leach, J., Mortimer, E., and Scott, P. (1994). Constructing scientific knowledge in the classroom. *Educational Researcher,* **23**(7), pp. 5–12.

Dubinsky, E. (1992*).* Reflective Abstraction in Advanced Mathematical Thinking. In D. Tall (Ed.) *Advanced Mathematical Thinking*. Kluwer: Dordrecht. pp. 95–126.

EFT (2014). *The qualifications of English and mathematics teachers Report prepared for the Education and Training Foundation (ETF)* [Accessed 112 January 2015]]<Available at : http://www.et-foundation.co.uk/wp-content/uploads/2014/09/RPT-Survey-v4.pdf>.

Eggen, P. D., and Kauchak, D.P. (2003). *Teaching and learning: research-based methods.* (4th Ed.) Allyn and Bacon.

Eisenhart, M.A., and Borko, H. (1991). In search of an interdisciplinary collaborative design for studying teacher education. *Teaching and Teacher Education,* **7**, pp. 135-157.

Eisner, E. W. (1991). *The enlightened eye: Qualitative inquiry and the enhancement of educational practice.* New York, NY: Macmillan Publishing Company.

Ekstam, U., Korhonen, J., Linnanmäki, K., and Aunio, P. (2018). Special education and subject teachers' self-perceived readiness to teach mathematics to low-performing middle school students. *Journal of Research in Special Educational Needs*, **18**(1), pp. 59-69.

Elden, M., and Chisholm, R. F. (1993). Emerging varieties of action research: Introduction to the special issue. *Human relations*, **46**(2), pp. 121-142.

Ellerbruch, L. W., and Payne, J. N. (1978). A teaching sequence for intial fraction concepts through the addition of unlike fractions. In Suydam, M. (Ed) *Developing computational Skills.* Reston. Virgina. National Council off Teachers of Mathematics.

Ellerton, N. F., and Clarkson, P. C. (1996). Language factors in mathematics teaching and learning. In A. J. Bishop, A.J., M. A. Clements,M. A C. Keitel, J. Kilpatrick and C. Laborde (Eds.), *International handbook of mathematics education.* Springer. pp. 987-1033.

Ellerton, N., and Clements, M. (1990). Cognitive dissonance versus success as the basis for meaningful mathematical learning. *Paper presented at the Proceedings of the 14th Conference of the International Group for the Psychology of Mathematics Education*, **1**, pp. 177-184.

Ellerton, S. (1990). An emerging model of the language of mathematics. In Bickermore-Brand, J. (Ed.), *Language in mathematics.* Carlton, Vic.: Australian Reading Association. pp. 27-42.

Elliott, J. (1991). *Action research for educational change.* Milton Keynes: Open University Press.

Ellis, V. (2007). *Learning and teaching in secondary schools* (3rd Ed.) Learning Matters.

Enderson, M.C., Grant, M.R., and Liu, Y. (2018). Supporting Mathematics Coaches' Learning of Probability through Professional Development Tasks. *International Journal of Research in Education and Science,* **4**(2), pp. 613-632.

Erickson, F. (1992). Ethnographic microanalysis of interaction. In M. D. LeCompte, W. L. Millroy, and J. Preissle (Eds.), *The handbook of qualitative research in education.* London: Academic Press. pp. 201–225.

Erickson, F. (2006). Definition and analysis of data from videotape: Some research procedures and their rationales. In J. Green, J., Camilli, G., and Ellmore, P. (Eds.), *Handbook of complementary methods in education research.* Mahwah, NJ: Lawrence Erlbaum Associates. pp. 177-191.

Ericsson, K. A. (2016). Summing up hours of any type of practice versus identifying optimal practice activities: Commentary on Macnamara, Moreau, and Hambrick (2016)*. Perspectives on Psychological Science,* **11**(3), pp. 351–354.

Ernest, P. (1989). *The impact of beliefs on the teaching of mathematics. Mathematics Teaching: The State of the Art*. The Falmer Press, Taylor and Francis, Inc. Bristol.

Ernest, P. (1989a). The knowledge, beliefs and attitudes of the mathematics teacher: A model. *Journal of Education for Teaching,* **5**(1), pp. 13-33.

Ernest, P. (1991). *The Philosophy of Mathematics Education.* London: Falmer Press.

Ernest, P. (1993). Dialectics in mathematics. Paper presented at 19th international conference on the history of science. Zaragoza, Spain, August. (A revised version is: The dialogical nature of mathematics). In Ernest ,P.(Ed.), *Mathematics, education and philosophy: An international perspective.* London: The Falmer Press.

Ernest, P. (1995). Values, Gender and Images of Mathematics: A Philosophical Perspective. *International Journal of Mathematical Education in Science and Technology,* **26**(3), pp. 449-462.

Ernest, P. (1996*).* Popularisation: Myths, Massmedia and Modernism. In Bishop, A. (Ed.), *International Handbook of Research in Mathematics Education.* Dordrecht, Holland. Kluwer.

Ernest, P. (2000).'Why teach mathematics?' In S. Bramall and J White, *Why learn mathematics?* London: Institute of Education (Bedford Papers Series).

Ernest, P. (2002*).* Empowerment in mathematics education. *Philosophy of Mathematics Education Journal*, **15**(1), pp.1-16.

Ernest, P. (2013). *Philosophy of Mathematics Education*. Routledge.

Ernest, P. (2016). Mathematics and values. In *Mathematical cultures*. Birkhäuser, Cham. pp. 189-214.

Ertmer, P. A., and Newby, T. J. (1993). Behaviorism, cognitivism, constructivism: Comparing critical features from an instructional design perspective. Performance improvement quarterly, **6**(4), pp. 50-72.

Etherington, K. (2007). Ethical research in reflexive relationships. *Qualitative inquiry*, **13**(5), pp. 599-616.

Evans, H. (1999). Learning to teach; learning from teaching. *Dimensions of Teaching and Learning,* pp. 3-28.

Evans, N. (1994). *Experiential learning for all.* London, New York: Cassell.

Fain, J. (2004). *Reading understanding and applying nursing research.* Second Edition. F. A. Davis Company, Philadelphia.

Fasko, D. (2016). Commentary on Uses of arguments from definition in children's argumentation. OSSA Conference Archive. 36. [Accessed : 31 December 2018] < Available at : https://scholar.uwindsor.ca/ossaarchive/OSSA11/papersandcommentaries/36>

Fasse, B., and Kolodner, J.L. (2000). Evaluating Classroom Practices Using Qualitative Research Methods: Defining and Refining the Process. In Fishman, B. and O'Connor-Divelbiss, S. (Eds.), *Fourth International Conference of the Learning Sciences*. Mahwah, NJ: Erlbaum. pp. 193-198.

Feiman-Nemser, S. (1983). Learning to teach. In Shulman, L.S. and Sykes G. (Eds.), *Handbook of teaching and policy.* New York: Longman.

Fennema, E., and Franke, M. L. (1992). Teachers' knowledge and its impact. In Grouws, D.A. (Ed.), *Handbook of research on mathematics teaching and learning*. New York: Macmillan. pp. 147-164.

Feucht, F. C., Lunn Brownlee, J., and Schraw, G. (2017). Moving beyond reflection: Reflexivity and epistemic cognition in taching and teacher education. Educational Psychologist, **52**(4), pp. 234–241.

Fitzgerald, A., Hackling, M., and Dawson, V. (2013). Through the viewfinder: Reflecting on the collection and analysis of classroom video data. *International Journal of Qualitative Methods, **12***, pp. 52–64.

Fitzsimons, G. E., Jungwirth, H., Maab, J., and Schloeglmann, W. (1996). Adults and mathematics (Adult numeracy). In Bishop, A. J. (Ed.), *The International Handbook of Mathematics Education.* Dordrecht: Kluwer Academic . pp. 755-784.

Flick, U. (2014). *An introduction to qualitative research (5th Ed.).* London: Sage Publications Ltd.

Flores, A. (2002). How do children know that what they learn in mathematics is true? *Teaching Children Mathematics,* **8**(5), pp. 269-274.

Flores, A. (2006*).* How do students know what they learn in middle school mathematics is true? *School Science and Mathematics,* **106**(3), pp. 124-132.

Flutter, J., and Rudduck, J. (2004). *Consulting pupils: What's in it for schools?* London: Routledge Falmer.

Flyvbjerg, B. (2011). Case study. In Denzin, N. K., and Lincoln, Y. S. (Eds.), *The Sage handbook of qualitative research* (4th ed.). Thousand Oaks, CA: Sage. pp. 301-316.

Foote, M. Q., and Bartell, T. G. (2011). Pathways to equity in mathematics education: How life experiences impact researcher positionality. *Educational Studies in Mathematics*, **78**(1), pp. 45-68.

Foote, M. Q., and Kim, S. U. (2015). *Introduction - Instructional support for language barriers in the learning and teaching of mathematics.* Reading and Writing Quarterly Overcoming Learning Difficulties, **31**(3), pp. 201-206.

Fosnot, C. T., and Dolk, M. (2002). *Young mathematicians at work: Constructing fractions, decimals, and percents.* Portsmouth, N.H.: Heinemann Press.

Foster, C. (2013c). Resisting reductionism in mathematics pedagogy. *Curriculum Journal*, **24**(4), pp. 563–585.

Foster, C. (2017). *Developing mathematical fluency: Comparing exercises and rich tasks.* Educational Studies in Mathematics, pp. 1-21.

Foster, C., and Inglis, M. (2018). Mathematics Teacher Professional Journals: What Topics Appear and How Has This Changed over Time? *International Journal of Science and Mathematics Education*, pp. 1-22.

Fox, R. W. (1981). *Basic skills in mathematics:* Book 2. London: Pitman Press.

Francis, L. J. (1992). Primary school size and pupil attitudes: Small is happy? *Educational Management and Administration,* **20**(2), pp. 100-104.

Frank, M. L. (1990). What Myths about Mathematics are held and conveyed by teachers? *Arithmetic Teacher*, **37**(1), pp. 10-12.

Froelich, J. (2009). Effective lesson design: A basic conceptual outline. *Using Looking at Learning to Improve School Performance,* pp. 1-5.

Fuchs, L. S., Powell, S. R., Seethaler, P. M., Cirino, P. T., Fletcher, J. M., Fuchs, D., and Hamlett, C. L. (2010). The effects of strategic counting instruction, with and without deliberate practice, on number combination skill among students with mathematics difficulties. *Learning and Individual Differences,* **20**, pp. 89–100.

Fuller, R. A. (1997). Elementary teachers' pedagogical content knowledge of mathematics. *Mid-Western Educational Researcher*, **10**(2), pp. 9-16.

Fulton S, Krainovich-Miller B (2010). Gathering and appraising the literature. In LoBiondo-Wood, G. and Haber J (Eds.), *Nursing research: methods and critical appraisal for evidence based practice.* Seventh Edition. Mosby Elsevier, Missouri, pp. 56 – 84.

Furner, J. M., and Worrell, N.L. (2017). "The Importance of Using Manipulatives in Teaching Math Today." *Transformations,* **3**(1), Article 2. [Accessed 31 December 2017] < Available at: http://nsuworks.nova.edu/transformations/vol3/iss1/2>.

Fusch, P. I., and Ness, L. R. (2015). Are we there yet? Data saturation in qualitative research. *The qualitative report*, **20**(9), pp. 1408-1416.

Gabriel, F.C., Coché, F., Szucs, D., Carette, V., Rey, B. and Content, A. (2013). A componential view of children's difficulties in learning fractions. *Frontiers in psychology*, *4*, p. 715.

Galton, M. J., and Patrick, H. (1990). *Curriculum provision in the small primary school.* Taylor and Francis.

Garaway, G. B. (1994). Language, culture, and attitude in mathematics and science learning: A review of the literature. *Journal of Research and Development in Education,* **27**(2), pp.102-111.

Gardiner, T. (2016). *Teaching mathematics at secondary level*. Open Book Publishers.

Gawned, S. (1990). An emerging model of the language of mathematics. *Language in mathematics*, pp. 27-42.

Gelman, R., Meck, E., and Merkin, S. (1986). Young children’s numerical competence. *Cognitive Development*, **1**, pp. 1–29.

Gerofsky, S., Sinclair, N. and Davis ,B. (2002). Mathematics and the Arts. In Elaine Simmt,. E and Davis. D, A. (Eds.), *Proceedings of the ACTES Annual meeting.* Queen’s university. Ontario Canada.

Gewirtz, S., and Cribb, A. (2006) What to do about values in social research: the case for ethical reflexivity in the sociology of education. *British Journal of Sociology of Education*, **27**(2), pp. 141-156.

Ghaith, G., and Yaghi, H. (1997*).* Relationships among experience, teacher efficacy, and attitudes toward the implementation of instructional innovation. *Teaching and Teacher Education*, **13**(4), pp. 451-458.

Ghamrawi, N. (2010). *No Teacher Left Behind: Subject Leadership that Promotes Teacher Leadership. Educational Management Administration and Leadership*, **38**(3), pp. 304–320.

Ginsburg, A., Leinwand, S., Anstrom, T., and Pollock, E. (2005). *What the united states can learn from Singapore’s world-class mathematics system (and what*

Singapore can learn from the united states): An exploratory study. American Institutes for Research.

Gobo, G. (2008). *Doing ethnography*. London, United Kingdom: Sage.

Goldhaber, D. D., and Brewer, D. J. (2000). Does teacher certification matter? high school teacher certification status and student achievement. *Educational Evaluation and Policy Analysis,* **22**(2), pp.129-145.

Goos, M., Galbraith, P. and Renshaw, P. (2004). Establishing a community of practice in a secondary mathematics classroom. Mathematics education: Exploring the culture of learning, pp. 36-61. In Burton, L. (Ed.), (2012). *Learning mathematics: From hierarchies to networks.* Routledge.

Gordon, L. M. (2001). High teacher efficacy as a marker of teacher effectiveness in the domain of classroom management. *Paper presented at the annual meeting of the California Council on Teacher Education.* San Diego, CA. (ERIC Document 224 Reproduction Service No. ED465731).

Goulding, M., Rowland, T., and Barber, P. (2002*).* Does it matter? Primary teacher trainees' subject knowledge in mathematics. *British Educational Research Journal*, **28**(5), pp. 689–704.

Governors (2015). *How many science and maths teachers have qualifications relevant to the subjects they teach?* [Accessed 7 April 2015] <Available at : http://www.questionsforgovernors.co.uk/teaching/how-many-science-and-maths-teachers-have-qualifications-relevant-to-the-subjects-they-teach/ >.

Grant, F., and Searl, J. (1997). Practical activities in the mathematics classroom. *Mathematics in School,* **26**(4), pp. 27-28.

Greabell, L.C. (1978). The effect of stimuli input on the acquisition of introductory geometry concepts by elementary school children*. School Science and Mathematics*, **78,** pp. 320–326.

Greaves, A. (2014). Supply of specialist subject teachers in secondary schools. *Report to Parliament 2: 2013-14.* Brisbane: Queensland Audit Office.

Greeno, J. G., Collins, A. M., and Resnick, L. B. (1996). Cognition and learning. In Berliner, D., and Calfee, R. (Eds.), *Handbook of educational psychology*. New York: Macmillan. pp. 15–46.

Greenwood, D. J., and Levin, M. (1998). Action research, science, and the co-optation of social research*. Studies in Cultures, Organizations and Societies,* **4**(2), pp. 237-261.

Greenwood, D. J., and Levin, M. (2006). *Introduction to action research.* Social research for social change SAGE publications.

Groundwater-Smith, S., and Mockler, N. (2007). Ethics in practitioner research: An issue of quality. *Research Papers in Education*, **22**(2), pp. 199-211.

Grouws, D. (2006). *Handbook of research on mathematics teaching and learning:(A project of the national council of teachers of mathematics)*. IAP.

Guba, E. G., and Lincoln, Y. S. (1994*).* Competing paradigms in qualitative research. *Handbook of Qualitative Research,* **2**(1), pp.105-117.

Guillemin, M., and Gillam, L. (2004). Ethics, reflexivity, and "ethically important moments" in research. *Qualitative Inquiry,* **10**(2), pp. 261-280.

Guishard, M. A., Halkovic, A., Galletta, A., and Li, P. (2018). Toward Epistemological Ethics: Centering Communities and Social Justice in Qualitative Research. *Forum Qualitative Sozialforschung / Forum: Qualitative Social Research*, **19**(3), pp. 1-24.

Guyton, E., and Farokhi, E. (1987). Relationships among academic performance, basic skills, subject matter knowledge, and teaching skills of teacher education graduates. *Journal of Teacher Education,* **38**(5), pp. 37-42.

Halliday, M.A.K. (1978). *Language as social semiotic.* London: Edward Arnold.

Hammersley, M. (1987). Some notes on the terms 'validity' and 'reliability'. *British Educational Research Journal*, **13**(1), pp. 73-82.

Hammersley, M. (2002). *Educational Research, Policymaking and Practice*. London : Paul Chapman Publishing.

Hamza, K., Piqueras, J., Wickman, P. O, and Angelin, M. (2018). Who owns the content and who runs the risk? Dynamics of teacher change in teacher–researcher collaboration. *Research in science education*, **48**(5), pp. 963-987.

Handerson, D. (1981). Three papers*. For the Learning of Mathematics,* **1**(3), pp. 12-15.

Hansen, P. J. K. (2009). The most important single factor influencing learning is what the learner already knows-What do the learner know about clouds, precipitation, wind and greenhouse effect; a short review of research from 1883 to 2009. In *9th EMS Annual Meeting, 9th European Conference on Applications of Meteorology (ECAM).* Toulouse, France.

Harel, G., and Sowder, L. (1998*). Students' proof schemes: Results from exploratory studies. Research in Collegiate Mathematics Education III*, pp. 234-283.

Harel, G., and Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. *Second Handbook of Research on Mathematics Teaching and Learning*, **2**, pp. 805-842.

Harris, A. (1999). *Teaching and learning in the effective school*. London: Arena Press.

Harris, A. (2013). *Effective subject leadership in secondary schools: A handbook of staff development activities.* London: David Fulton Publishers.

Harris, J. B., Hofer, M. J., Schmidt, D.A., Blanchard, M. R., Young, C.Y., Grandgenett, N.F., van Olphen, M. (2010*).* “Grounded” Technology Integration: Instructional Planning Using Curriculum-Based Activity Type Taxonomies. *Journal of Technology and Teacher Education*, **18**(4), pp. 573-605.

Harrison, J., Dymoke, S., and Pell, T. (2006). Mentoring beginning teachers in secondary schools: An analysis of practice. *Teaching and Teacher Education*, **22**(8), pp. 1055-1067.

Harry, B., and Lipsky, M. (2014). Qualitative Research on Special Education Teacher Preparation. In M. McCray, T. Brownell, & B. Lignugaris/Kraft (Eds.), *Handbook of research on special education teacher preparation*, pp. 445-460.

Hartshorn, R., and Boren, S. (1990). Experiential learning of mathematics: Using manipulatives*. ERIC Clearinghouse on Rural Education and Small Schools*. ED321967.

Hatch, T., and Gardner, H. (1990). If Binet had looked beyond the classroom: The assessment of multiple intelligences*. International Journal of Educational Research,* **14**(5), pp. 415-429.

Hattie, J. (2003). *Teachers make a difference, what is the research evidence?* Australian Council for Educational Research: Annual Conference on Building Teacher Quality. Melbourne.

Hattie, J. (2012). *Visible Learning for Teachers: Maximizing Impact on Learning.* Routledge.

Hauk, S., Jackson, B., and Tsay, J.J. (2017*).* Those who teach the teachers: Knowledge growth in teaching for mathematics teacher educators. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, S. Brown (Eds.), *Proceedings of the 20th Conference on Research in Undergraduate Mathematics Education,* San Diego, CA.

Healy, M., and Perry, C. (2000). Comprehensive criteria to judge validity and reliability of qualitative research within the realism paradigm. *Qualitative Market Research: An International Journal*, **3**(3), pp. 118-126.

Heath, C., Hindmarsh, J., and Luff, P. (2010). *Video in qualitative research: Analysing social interaction in everyday life.* London, United Kingdom: Sage.

Henson (2013). Skills in Teaching Mathematics. In Brumbaugh, D.K, and Rock, D (2013). *Teaching Secondary Mathematics, 3rd edition,* New Jersey London.: Lawrence Erlbaum Associates.

Hersh, R. (1997). "Math Lingo vs. Plain English Double Entendre." *American Mathematical Monthly,* **104**, pp. 48-51.

Hershkowitz, R., and Schwarz, B. B. (1999). 'Reflective processes in a technology-based mathematics classroom'. *Cognition and Instruction,* **17**, pp. 66–91.

Hevern, V. (2003). *Philosophical perspective. Narrative Psychology: Internet and Resource Guide*. [Accessed March 1, 2014] <Available at : from the Le Moyne College Web Site: http://web.Lemoyne.Edu/-Hevednr-Constr.htm1>.

Hewitt, D. (2015). The economic use of time and effort in the teaching and learning of mathematics. In S. Oesterle & D. Allan (Eds.), *Proceedings of the 2014 Annual Meeting of the Canadian Mathematics Education Study Group*, Edmonton, Canada. pp. 3-23

Hiebert, J., and Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. *Second Handbook of Research on Mathematics Teaching and Learning,* **1**, pp. 371-404.

Hiebert, J., and LeFevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics.* Hillsdale: Erlbaum. pp. 1-27.

Hill, H. C., Ball, D. L., and Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, pp. 372-400.

Hill, H. C., Rowan, B., and Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, **42**(2), pp. 371-406.

Hilton, G. L. (2017). Part 3 education policy, reforms and school leadership. *Current Business and Economics Driven Discourse and Education: Perspectives from Around the World,* pp. 79-85.

HMSO (1985a). *The Curriculum from 5 to 16. Curriculum Matters 2.* An HMI series, DES.

HMSO (1985b). The *Mathematics from 5-16. Curriculum Matters 3.* An HMI Series, DES.

Hoffert, S. B. (2009). Mathematics: The universal language? *Mathematics Teacher*, **103**(2), pp. 130-139.

Hoffman, C. (1991). *An introduction to Bilingualism*, Longman Linguistic Library

Hofmann, R., and Mercer, N. (2016).Teacher interventions in small group work in secondary mathematics and science lessons. *Language and Education,* **30**(5), pp. 400 – 416.

Hooks, B. (2003). *Teaching community : A pedagogy of hope*. New York: Routledge.

Hopkins, D., Ainscow, M., and West, M. (1997). Unravelling the complexities of school improvement: A case study of the improving the quality of education for all (IQEA) project. course E838 reader. In Harris, A., Bennett, N. and Preedy, M. (Eds.), *Organizational effectiveness and improvement*. Milton Keynes: Open University Press.

Howard, A. J. (2018). A Descriptive Study of High School Mathematics Teachers' Perceptions of Flipped-Mastery Learning in a High School Math Class.

Hoyles, C. (1982). The pupil's view of mathematics learning. *Educational Studies in Mathematics,* **13**(4), pp. 349-372.

Hunting, R.P. (2003). Part-whole number knowledge in preschool children. *Journal of Mathematical Behavior*, **22**(3), pp. 217-235.

Hyde, J. S., and Jafeee, S. (1998). Perspectives from social and feminist psychology. *Educational Researcher*, **27**(5), pp. 14-16.

ICMI (2013). *Proceedings of the international Conference for ICMI Study 22*, Oxford, UK, Task Design in Mathematics Education. [Accessed 1 November 2016] <Available at: https://hal.archives-ouvertes.fr/hal-00834054 >.

Illeris, K. (2009). *Contemporary theories of learning: Learning theorists -- in their own words*. London: Routledge.

Illeris, K. (2018). A comprehensive understanding of human learning. In *Contemporary theories of learning.* Routledge. pp.15-28.

Ingram, J., and Ward-Penny, R. (2010). Otherwise engaged. *Mathematics Teaching,* **217**, p.18.

Ingram, J., Sammons, P., and Lindorff, A. (2018). Observing Effective Mathematics Teaching: A Review of the Literature. *Education Development Trust*.

Inouye, M. (2017). The role of secondary STEM teacher professional development in integrating critical thinking skills through teacher perception and classroom observation. *SMTC Plan B Papers*. **52**.

Irvine, J. (2017). "Problem Posing in Consumer Mathematics Classes: Not Just for Future Mathematicians". *The Mathematics Enthusiast*, **14**(1), p. 387- 411.

Jacob, R., Hill, H., and Corey, D. (2017). The impact of a professional development program on teachers' mathematical knowledge for teaching, instruction, and student achievement. *Journal of Research on Educational Effectiveness,* **10**(2), pp. 379-409.

Jamieson, S. (2004). Likert scales: how to (ab)use them. *Medical Education,* **38**, pp. 1212–1218.

Jarvis, P., Holford, J., and Griffin, C. (2003). The theory and practice of learning. *Psychology Press.*

Jaworski, B. (1994). I*nvestigating Mathematics Teaching: A Constructivist Enquiry.* London: Falmer Press.

Jaworski, B. (1996). *Constructivism and teaching-the socio-cultural context.* [Accessed October, 26, 2011] <Available at: http://www.grout.demon.co.uk/Barbara/chreods.htm>.

Jaworski, B. (2007). Learning communities in mathematics: Research and development in mathematics teaching and learning. *Paper presented at the Relating Practice and Research in Mathematics Education. Proceedings of Norma05, Fourth Nordic Conference on Mathematics Education.* Trondheim: Tapir Akademisk Forlag.

Jennings, J. L., and Bearak, J. M. (2014*).* "Teaching to the test" in the NCLB era: How test predictability affects our understanding of student performance. *Educational Researcher*, **43**(8), pp. 381-389.

Jewitt, C. (2012). *An introduction to using video for research.* National Centre for Research Methods Working Paper 03/12. [Accessed 1 May 2014] <Available at : http://eprints.ncrm.ac.uk/2259/4/NCRM_workingpaper_0312.pdf>.

Jick, T. D. (1983). Mixing qualitative and quantitative methods: Triangulation in action. In J. van Maanen (Ed.), *Qualitative methodology*. London: Sage Publications. pp. 135-148.

Jitendra, A. K., Star, J. R., Starosta, K., Leh, J. M., Sood, S., Caskie, G., Mack, T. R. (2009). Improving seventh grade students' learning of ratio and proportion: The role of schema-based instruction. *Contemporary Educational Psychology*, **34**(3), pp. 250-264.

John, P. D. (1991*).* "A Qualitative Study of British Student Teachers' Lesson Planning Perspectives". *Journal of Education for Teaching*, **17**(3), pp. 301-320

John, P. D. (1993). *Lesson Planning for Teachers.* London: Cassell.

John, P. D. (1994). "The Integration of Research Validated Knowledge with Practice: lesson planning and the student history teacher". *Cambridge Journal of Education*, **24**(1), pp. 33-47.

John, P. D. (2006). Lesson planning and the student teacher: Re-thinking the dominant model. *Journal of Curriculum Studies,* **38**(4), pp. 483–498.

Johnson, B., Peters, J., and Williams, D. (1999). Academics working with schools: Revealing the tensions. *Journal of Education for Teaching,* **25**(2), pp. 123-133.

Johnson, R. B. (1997). Examining the validity structure of qualitative research. *Education*, **118**(2), pp. 282-292.

Jones, K., and Edwards, A. (2011). Planning for mathematics teaching. In Johnston-Wilder, S., Johnston-Wilder, P., Pimm, D. and Lee, C. (Eds.), *Learning to Teach Mathematics in the Secondary School: A Companion to School Experience. 4th edition.* New York, Taylor and Francis pp. 70-91.

Jones, S. R., Torres, V., and Arminio, J. (2006). *Negotiating the complexities of qualitative research in higher education: Fundamental elements and issues.* New York, NY: Routledge.

Joppe, M. (2000). *The Research Process.* [Accessed 25 February 2014] <Available at http://www.ryerson.ca/~mjoppe/rp.htm>.

Joyce, B., and Showers, B. (1980). The coaching of teaching. Educational Leadership, **40**(10), pp. 4-10.

Kajander, A., and Lovric, M. (2009). Mathematics textbooks and their potential role in supporting misconceptions. *International Journal of Mathematical Education in Science and Technology*, **40**(2), pp. 173-181.

Kambouri, M. (2016). Investigating early years teachers' understanding and response to children's preconceptions. *European Early Childhood Education Research Journal*, **24**(6), pp. 907-927.

Kamina, P., and Iyer, N. N. (2009). *From concrete to abstract: Teaching for transfer of learning when using manipulatives. Nera Conference Proceedings.*

Kapucu, N. (2012). Classrooms as communities of practice: Designing and facilitating learning in a networked environment. *Journal of Public Affairs Education*, **18**(3), pp. 585-610

Karatas, I., Guven, B., Öztürk, Y., Arslan, S., and Gürsoy, K. (2017). Investigation of Pre-School Teachers' Beliefs about Mathematics Education in Terms of Their Experience and Structure of Their Education, *Eurasia Journal of Mathematics, Science and Technology Education,* **13**(3), pp. 973-689.

Karnieli-Miller, O., Strier, R., and Pessach, L. (2009). Power relations in qualitative research. *Quality Health Research,* **19**(2), pp. 279-89.

Kazemi, E., and Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *Journal of Education,* **189**, pp. 123 – 137.

Keay, J. (2009). Being influenced or being an influence: New Teachers' induction experiences. *European Physical Education Review,* **15**(2)**,** p. 225.

Khaldi, K. (2017). Quantitative, Qualitative or Mixed Research: Which Research Paradigm to Use? *Journal of Educational and Social Research*, ***7***(2), pp.15-24.

Kieren, T. E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In Lesh, R. (Ed.), *Number and* *measurement: Papers from a research workshop Columbus, OH: ERIC/SMEAC.* pp. 101-144.

Kieren, T. E. (1980). The rational number constructs - Its elements and mechanisms. In Kieren, T.E. (Ed.), *Recent research on number learning Columbus:* ERIC/SMEAC. pp. 125-150.

Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In Hiebert, J. and Behr, M. (Eds.), *Number concepts and operation in the middle grades.* Reston, VA: National Council of Teachers of Mathematics. pp. 53-92.

Kieren, T. E. (1990). *Children's mathematics/mathematics for children.* Transforming Children's Mathematics Education: International Perspectives. Hillsdale, NJ: Lawrence Erlbaum Associates.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In Carpenter, T. and Fennema, E. and Romberg, T. (Eds.), *Rational numbers: An integration of research.* Hillsdale, NJ: Lawrence Erlbaum Associates. pp. 49-84.

Kieren, T. E. (1994*).* Orthogonal reflections on computer microworlds, constructivism, play and mathematical understanding. *Journal for research in Childhood Education,* **8**, pp. 132-141.

Kilpatrick, J. (1987). Problem formulating: where do good problems come from? In A.H. Schoenfeld (Ed.), *Cognitive science and mathematics education.* Hillsdalc, NJ: Erlbaum. pp. 123-147.

Kilpatrick, J., Swafford, J., and Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Klein, H. K., and Myers, M. D. (1999). A set of principles for conducting and evaluating interpretive field studies in information systems. *MIS Quarterly*, **23**(1), pp. 67-93.

Kleiner, R. J., and Okeke, B. I. (1991*). Advances in field theory: New approaches and methods in cross-cultural research. Journal of Cross-Cultural Psychology*, **22**(4), pp. 509-524.

Kleitman, S., and Stankov, L. (2007). Self-confidence and metacognitive processes. *Learning and Individual Differences*, **17**(2), pp. 161-173.

Koehler, M. S., and Grouws, D. A. (1992). Mathematics teaching practices and their effects. *Handbook of Research on Mathematics Teaching and Learning,* **1**, pp.15-126.

Kolb, D. A. (2015). *Experiential learning: Experience as the source of learning and development (2nd Ed.).* Upper Saddle River: Pearson Education, Inc.

Kooken, J., Welsh, M., McCoach, B., Johnston-Wilder, S., and Lee, C. (2013). *Measuring Mathematical Resilience: An Application of the Construct of*

Resilience to the Study of Mathematics. AERA 2013, San Francisco, California.

Kotsopoulos, D. (2007). Mathematics Discourse It's like hearing a foreign Language*, NCTM,* **4**, pp. 301-305.

Kuhs, T. M., and Ball, D. L. (1986). Approaches to teaching mathematics: Mapping the domains of knowledge, skills, and dispositions. *East Lansing: Michigan State University, Center on Teacher Education*.

Kukla, A. (2013). *Social constructivism and the philosophy of science*. Routledge.

Kunter, M., Baumert, J., Blum, W., Klusmann, U., Krauss, S., and Neubrand, M. (Eds.) (2013). *Cognitive activation in the mathematics classroom and professional competence of teachers: Results from the COACTIV project*. Springer Science and Business Media. pp. 249-271.

Kvale, S. (1996). *Interviews : An introduction to Qualitative research interviews.* California: Sage Publications.

Kyriacou, C. (1992). Active learning in secondary school mathematics. *British Educational Research Journal,* **18**(3), pp. 309-318.

Labaree, D. F. (2004). *The trouble with ed schools*. New Haven, CT: Yale University Press.

Lam, R. (2015). Language assessment training in Hong Kong: Implications for language assessment literacy*. Language Testing*, **32**(2), pp. 169-197.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching*. American Educational Research Journal*, **27**, pp. 29-63.

Lampert, M., Beasley, H., Ghousseini, H., Kazemi, E., and Franke, M. (2010). Using designed instructional activities to enable novices to manage ambitious mathematics teaching. *Instructional explanations in the disciplines.* Springer. pp. 129-141.

Lather, P. (1986). Research as praxis. *Harvard Educational Review*, **56**(3), pp. 257-278.

Lather, P. (2003). Issues of validity in openly ideological research: Between a rock and a soft place. In Y. S. Lincoln and N. K. Denzin (Eds.), *Turning points in qualitative research: Tying knots in a handkerchief.* Walnut Creek, CA: AltaMira Press.

Lau, K.H., Lam, T., Kam, B.H., Nkhoma, M.,Richardson, J., and Thomas, S. (2018).The role of textbook learning resources in e-learning: A taxonomic study. *Computers and Education,* **118**, pp. 10-24.

Lave, J. (1988). *Cognition in Practice: Mind, mathematics, and culture in everyday life.* Cambridge, UK: Cambridge University Press.

Lave, J., and Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge university press.

LeCompte, M. D., and Goetz, J. P. (1982). Problems of reliability and validity in ethnographic research. Review of Educational Research, 52(1), pp. 31-60.

Lederman, N. G., and Gess-Newsome, J. (1992). Do subject matter knowledge, pedagogical knowledge, and pedagogical content knowledge constitute the ideal gas law of science teaching? Journal of Science Teacher Education, 3(1), pp. 16-20.

Lee, C. (2006). Language For Learning Mathematics: Assessment For Learning In Practice: Assessment for Learning in Practice. McGraw-Hill Education (UK).

Lee, C., and Johnston-Wilder, S. (2013). Learning mathematics—letting the pupils have their say. *Educational Studies in Mathematics*, **83**(2), pp. 163-180.

Lee, C., and Johnston-Wilder, S. (2015). Mathematical resilience. What is it and why is it important? In S. Chinn (Ed.), International Handbook of Dyscalculia and Mathematical Learning Difficulties (pp. 337- 345). London: Routledge.

Leedy, P. D., and Ormrod, J. E. (2010). *Practical research : Planning and Design (9th ed.)*. Saddle River, New Jersey: Pearson Education.

Lehtinen, E., Hannula-Sormunen, M., McMullen, J., and Gruber, H. (2017). Cultivating mathematical skills: from drill-and-practice to deliberate practice. *ZD*M Mathematics Education, **49**(4), pp. 625-636.

Leinhardt, G., and Smith, D. A. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology,* **77**(3), p. 247-271.

Leinhardt, G., Zaslavsky, O., and Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, **60**, pp. 1–64.

Lemke, J.L. (1982). Talking physics*. Physics education (UK)*, **17**, pp. 263- 267.

Leont'ev, A. A. (1975). *Dieyatelinocti, soznaine, i lichynosti (activity, consciousness, ans personality).* Moskva: Politizdat.

Lerman, S. (1990). Alternative perspectives of the nature of mathematics and their influence on the teaching of mathematics. *British Educational Research Journal,* **16**(1), pp. 53-61.

Lerman, S. (1999). Some problems in research on mathematics teaching and learning from a socio-cultural approach*. Informal Proceedings 16-3 (BSRLM),* pp. 33-36.

Lerman, S., and Cowley, R. (2012) Learning theories in the mathematics classroom. In S. Johnston Wilder and C. Lee (Eds.), *Leading Practice and*

Managing Change in the Mathematics Department. St. Albans. Tarquin Publications

Levitt, H. M. (2015). Interpretation-Driven Guidelines for Designing and Evaluating Grounded Theory Research: A Constructivist-Social Justice Approach. In O. C. G. Gelo, A. Pritz, and B. Rieken (Eds.), *Psychotherapy Research*. Springer Vienna pp. 455–483.

Lewis, C. C., Perry, R. R., and Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and North American case. *Journal of Mathematics Teacher Education*, **12**(4), pp. 285-304.

Lewis, G. (2014). The incidence of disaffection with school mathematics. In *Proceedings of the Joint Meeting of PME,* **38**, pp. 121-128.

Li, Y., Zhao, D., Huang, R., and Ma, Y. (2008). Mathematical preparation of elementary teachers in china: Changes and issues. *Journal of Mathematics Teacher Education,* **11**(5), pp. 417- 430.

Light, P. (2017). Social interaction and cognitive development: a review of post-Piagetian research. In Meadows, S. (Ed.), *Developing Thinking : Approaches to Children's Cognitive Development*, Routledge, pp. 67-88.

Liljedahl, P. (2010). Noticing rapid and profound mathematics teacher change. *Journal of Mathematics Teacher Education,* **13**(5), pp. 411-423.

Liljedahl, P., Rösken, B., and Rolka, K. (2006*).* Documenting changes in pre-service elementary school teachers' beliefs: Attending to different aspects. *Paper presented at the Proceedings of the 28th International Conference for Psychology of Mathematics Education–North American Chapter 2*. pp. 279-285.

Lincoln, Y. S., and Guba, E. G. (1985). *Naturalistic inquiry.* Beverly Hills: Sage.

Lithner, J. (2003), 'Students' Mathematical Reasoning in University Textbooks Exercises'. *Educational Studies in Mathematics,* **52**, pp. 29–55.

Littlewood, W. (2004). The task-based approach: Some questions and suggestions. *ELT Journal*, **58**(4), pp. 319-326.

Lochhead, J., and Mestre, J. P. (1988). From words to algebra: Mending misconceptions. In A. F. Coxford and A. P. Shulte (Eds.),*The ideas of algebra, K–12: The 1988 yearbook of the National Council of Teachers of Mathematics.* Reston, VA. pp. 127–135.

Loewenberg Ball, D., Thames, M. H., and Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education,* **59**(5), pp. 389-407.

Loong, Y. (2014). Fostering mathematical understanding through physical and virtual manipulatives. *The Australian Mathematics Teacher,* **70**(4), pp. 3-10.

Lortie, D. (1975). Schoolteacher: A sociological study*. Chicago: University of Chicago*.

Love, A., and Kruger, A. C. (2005). Teacher beliefs and student achievement in urban schools serving African American students. *The Journal of Educational Research*, **99**(2), pp. 87-98.

Löwe, B., and Müller, T. (2010). Skills and mathematical knowledge. In B. Löwe, and T. Müller (Eds.), PhiMSAMP*: philosophy of mathematics: sociological aspects and mathematical practice* (Texts in philosophy; No. 11). London: College Publications. pp. 265-280.

Lubienski, S. T., Lubienski, C., and Crane, C. C. (2008). Achievement differences and school type: The role of school climate, teacher certification, and instruction. *American Journal of Education,* **115**(1), pp. 97-138.

Luttrell, W. (2010) Introduction: The Promise of Qualitative Research in Education. *in* Luttrell, W. (Ed.), *Qualitative educational research: readings in reflexive methodology and transformative practice.* New York: Routledge, pp.1-17.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in china and the United States*. Lawrence Erlbaum Associates Mahwah, NJ.

Maaß, K. (2018). Scaling up Innovative Teaching Approaches in Mathematics: Supporting Teachers to take up a New Role as Professional Development Course Leaders for Inquiry-Based Learning. *Journal of Education and Training Studies*, **6**(7), pp.1-16.

MacDonald, B., and Walker, R. (1975). Case Study and the social philosophy of educational research. *Cambridge Journal of Education*, **5**(1), pp. 2-11.

Magus, W. (1997). The Significance of Mathematics: The Mathematicians' Share in the General Human Condition. *American Mathematical Monthly,* **104**, pp. 261-269.

Maher, C. (1991). Is dealing with mathematics as a thoughtful subject compatible with Maintaining satisfactory test scores? A nine-year study. *Journal of Mathematical Behavior,* **10**, pp. 225-248.

Major, C. H., and Savin-Baden, M. (2010). *An introduction to qualitative research synthesis: Managing the information explosion in social science research.* Routledge.

Manias, E., and McNamara, T. (2015). Standard setting in specific-purpose language testing: What can a qualitative study add? *Language Testing,* **33**(2), pp. 235-249.

Mann, E. L. (2006*). Creativity: The essence of mathematics. Journal for the Education of the Gifted,* **30**(2), pp. 236-260.

Manouchehri, A. (1997). School mathematics reform: Implications for mathematics teacher preparation. *Journal of Teacher Education*, **48**(3), pp.197-209.

Margolinas, C. (2013). *Task design in mathematics education. proceedings of ICMI study 22*. ICMI Study 22.

Marsh, L. G., and Cooke, N. L. (1996). The effects of using manipulatives in teaching math problem solving to students with learning disabilities*. Learning Disabilities Research and Practice,* **11**(1), pp. 58-65.

Marshall, C., and Rossman, G. B. (1999). *Designing qualitative research (3 Ed.).* Thousand Oaks, CA: Sage Publications.

Marzano, R. (2013). *The Critical Importance of a Common Language of Instruction.* Learning Sciences International, Blairsville, PA

Mason, J. (1991). Mathematical problem solving: Open, closed and exploratory in the UK. *ZDM,* **91**(1), pp. 14-19.

Mason, J., and Johnston-Wilder, S. (2006). *Designing and using mathematical tasks*. Tarquin publication. The Open University Press.

Masters, G. N. (2009). *A shared challenge: Improving literacy, numeracy and science learning in Queensland primary schools.* Camberwell, Victoria: Australian Council for Educational Research.

Matthews, C. (2014). Teaching division. The importance of coherence in what is made available to learn. In Venkat, H., Rollnick, M., Loughran, J. and Askew, M. (Eds.), *Exploring Mathematics and Science Teachers' Knowledge Windows into teacher thinking* Routledge. UK.

May, T. (2001). *Social Research: Issues, Methods and Process.* McGraw Hill. London.

McBer, H. (2001). Research into teacher effectiveness. *Early Professional Development of Teachers*, **68**, pp.1-69.

McClain, K., and Cobb, P. (2001). An analysis of development of sociomathematical norms in one first-grade classroom. *Journal for Research in Mathematics Education,* **32**(3), pp. 236-266.

McKernan, J. (2013). *Curriculum action research: A handbook of methods and resources for the reflective practitioner*. Routledge.

McLeod, D. B. (1992). Research on affect in mathematics education: A conceptualisation. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*. New York: Macmillan. pp. 575-596.

McMahon, M. (1997). Social constructivism and the World Wide Web-A paradigm for learning. In *ASCILITE conference. Perth, Australia* (Vol. 327).

McNamara, O., Jaworski, B., Rowland, T., Hodgen, J., and Prestage, S. (2002). *Developing mathematics teaching and teachers.* Unpublished Manuscript.

McNiff, J. (2013). *Action research: Principles and practice*. Routledge.

McNiff, J., and Whitehead, J. (2011). *All You Need to Know About Action Research*. London. Sage.

Meira, L. (1998). 'Making sense of instructional devices: The emergence of transparency in mathematical activity'*. Journal for Research in Mathematics Education,* **29**(2), pp. 121–142.

Mendiburo, M., and Hasselbring, T. (2011). *Technology's impact on fraction learning: An experimental comparison of virtual and physical manipulatives.* Paper presented at the Society for Research on Educational Effectiveness, Washington, DC.

Mercer, N. (2000). *Words and Minds: How we use language to think together.* London: Routledge.

Mercer, N., and Sams, C. (2006). Teaching children to use language to solve maths problems. *Language and Education,* **20**(6), pp. 507-528.

Merriam, S. B. (1998). *Qualitative research and case study applications in education. Revised and expanded from" case study research in education"*. (ERIC Reproduction Service No. ED415771).

Merriam, S. B., Johnson-Bailey, J., Lee, M. Y., Kee, Y., Ntseane, G., and Muhamad, M. (2001). Power and positionality: Negotiating insider/outsider status within and across cultures. *International Journal of Lifelong Education*, *20*(5), 405-416.

Metcalfe, C., and Russell, S. (1997). *The role of subject leaders in monitoring the work of teachers in secondary schools: the quest for consistency?* Paper presented to the annual conference of the British Educational Research Association, York.

Meyer, B. (2013). Changing teachers practices from within classrooms: IPads as mediators for change*. In Society for Information Technology and Teacher Education International Conference*, Chesapeake, VA. Association for the Advancement of Computing in Education (AACE), pp. 3747-3756.

Midlands Mathematics Experiment (MME) (1963). *O level book 3*, Harrap, London.

Miles, M. B., and Huberman, A. M. (1984). Qualitative data analysis: A sourcebook of new methods. In *Qualitative data analysis: a sourcebook of new methods*. Sage publications.

Milford, J. (1998). Head teacher influence on classroom teachers" use of student-centred strategies. *Reflections,* **23**(3), pp. 21-23.

Mills, J. (2014). Methodology and methods. In Mills, J., and Birks, M. (Eds.), *Qualitative methodology: A practical guide*. Thousand Oaks, CA: Sage. pp. 31-47

Minsky, M. L. (1975). A framework for representing knowledge. In P. H. Winston (Ed.), *The Psychology of Computer Vision.* New York: McGraw Hill. pp. 211–277.

Mirza, A., and Hussain, N. (2018). Performing Below the Targeted Level: An Investigation into KS3 Pupils' Attitudes Towards Mathematics. *Journal of Education and Educational Development*, **5**(1), pp. 8 – 24.

Mohan, B. A. (2012). Qualitative research methods in second language assessment. In L. V. Hedges, R. Coe, and M. Waring (Eds.), *Research methods and methodologies in education.* Thousand Oaks, CA: Sage, pp. 752-767.

Monk, D. H., and King, J. A. (1994). Multilevel teacher resource effects in pupil performance in secondary mathematics and science: The case of teacher subject matter preparation. *Choices and Consequences: Contemporary Policy Issues in Education,* pp. 29-58.

Monsour, F. (2000). Fostering Resiliency Through Service Learning. *Wisconsin School News,* **54**(12), pp. 22-23.

Moon, J. (2009). *A handbook of reflective and experiential learning. Theory and Practice.* London: Routledge.

Moor, H., Halsey, K., Jones, M., Martin, K., Stott, A., Brown, C., and Harland, J. (2005). Professional Development for Teachers Early in Their Careers: An Evaluation of the Early Professional Development Pilot Scheme. Research Report RR613. *National Foundation for Educational Research*.

Moreno, G. A., and Rutledge, D. (2018). A response to strategies and tactics through participatory action research in a developmental mathematics course. *Educational Action Research*, **26**(3), pp. 420-438.

Morgan, C. (2017*).* From policy to practice: Discourses of mastery and "ability" in England. *Mathematics Education and Life at Times of Crisis. Proceedings of the Ninth International Mathematics Education and Society Conference*, pp. 717-727.

Morley, D. (1991*).* Resource analysis as action research: *Resource analysis research in developing countries.* Toronto: York University. pp. 1-16.

Morris, H. (2001*). Issues raised by testing trainee primary teachers' mathematical knowledge. Mathematics Teacher Education and Development*, **3**, pp. 37-47.

Morris, R. W. (1978). The role of language in learning mathematics. *Prospects,* **8**(1), pp. 73-81.

Morrison, M. (2012). Understanding Methodology. In Briggs, A.R.J., Coleman M. and Morrison M., (Eds.), *Research methods in educational leadership and management. 3rd Ed.* London: SAGE. p. 14-28.

Moseley, B. (2005). Students' early mathematical representation knowledge; the effects of emphasizing single or multiple perspectives of the rational number domain in problem solving. *Educational Studies in Mathematics,* **60**(1), pp. 37-69.

Mtetwa, D., Garofalo, J. (1989). Beliefs about Mathematics: An Overlooked Aspect of Student Difficulties. *Academic Therapy,* **24**(5), pp. 611-618.

Muijs, D. (2004). *Doing quantitative research in education with SPSS*. London: Sage

Muijs, D., and Reynolds, D. (2015). Teachers' beliefs and behaviors: What really matters? *The Journal of Classroom Interaction*, **50**(1), pp. 3-15.

Naidoo, D., and Parker, D. (2005). The implications of mathematics teachers' identities and official mathematics discourses for democratic access to mathematics. *Perspectives in Education,* **23**(1) pp. 53-67.

Nardi, E., and Steward, S. (2003). Is mathematics T.I.R.E.D? A profile of quiet disaffection in the secondary mathematics classroom. *British Educational Research Journal*, **29**(3), pp. 345–367.

National Audit Office (NAO) (2016). *Initial teacher training in England.* Briefing paper 6710. Report by National Audit Office for Department for Education House of Commons Library. [Accessed 1 December 2017] <Available at : http://dera.ioe.ac.uk/26354/2/SN06710_Redacted.pdf >.

National Council of Supervisors of Mathematics (NCSM) (1977). Position paper on basic mathematical skills*. Arithmetic Teacher,* **25**(1), pp. 19-22.

National Council of Teachers of Mathematics (NCTM) (2000). *Executive Summary Principles and Standards for School Mathematics.* [Accessed 8 May 2012] <Available at: https://www.nctm.org/uploadedFiles/Standards_and_Positions/PSSM_ExecutiveSummary.pdf >.

National Council of Teachers of Mathematics. (NCTM) (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA.

National Council of Teachers of Mathematics. Commission on Teaching Standards for School Mathematics. (NCTM) (1991). *Professional standards for teaching mathematics.* National Council of Teachers.

National Research Council and Mathematics Learning Study Committee.(NRC) (2001). *Adding it up: Helping children learn mathematics*. National Academies Press.

Naylor, S., and Keogh, B. (1999). Constructivism in classroom: Theory into practice. *Journal of Science Teacher Education,* **10**, pp. 93-106.

Nespor, J. (1987). The role of beliefs in the practice of teaching. *Journal of Curriculum Studies,* **19**(4), pp. 317-328.

Newman, I., and Benz, C. R. (1998). *Qualitative-quantitative research methodology: Exploring the interactive continuum.* SIU Press.

Newman, M. A. (1979). *Maths and the second language child.* In S Ferguson (Ed.), *Mathematics for the 80s.* Parkville: The Mathematics Association of Victoria, pp. 153-159.

Nisbet, S., and Warren, E. (2000). Primary school teachers' beliefs relating to mathematics, teaching and assessing mathematics and factors that influence these beliefs. *Mathematics Teacher Education and Development*, **2**, pp. 34-47.

Nolan, A., and Molla, T. (2017). Teacher confidence and professional capital. *Teaching and Teacher Education,* **62**, pp.10-18.

Norton, S. (2016). Primary mathematics teacher education encountering difficult terrain: A personal perspective. *International Journal for Mathematics Teaching and Learning,* **17**(2), pp 2-33.

Novillis-Larson, C. (1980). Locating proper fractions on number lines: Effect of length and equivalence. *School Science and Mathematics*, **80**(5), pp. 423-428.

Núñez, R. E., Edwards, L. D., and Matos, J. F. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics,* **39**(3), pp. 45-65.

Nurlu, Ö. (2017). Investigation of teachers' mathematics teaching self-efficacy. *International Electronic Journal of Elementary Education*, **8**(1), pp. 21-40.

O'Brien, B. C., Harris, I. B., Beckman, T. J., Reed, D. A., and Cook, D. A. (2014). Standards for Reporting Qualitative Research: *A Synthesis of Recommendations. Academic Medicine*, **89**(9), 1245–1251.

O'Brien, R. (2001). An overview of the methodological approach of action research. In Richardson R. (Ed.), *Theory and practice of action research.* joão pessoa, Brazil: Universidade federal da paraíba. English Version.

Ofsted (2009). *Understanding the Score.* HMSO. [Accessed 1 April 2013] <Available at http://dera.ioe.ac.uk/9459/2/A9RAB12_Redacted.pdf>.

Ofsted (2012). *Mathematics: made to measure.* HMSO [Accessed 1 April 2013] <Available at www.ofsted.gov.uk/resources/110159>.

Ofsted. (2012a). *Handbook for inspecting schools in England under section 5 of the education act 2005 (as amended) from September 2012.* HMSO.

Oldfield, R. C., and Zangwill, O. L. (1942a). Head’s concept of the schema and its application in contemporary British psychology. *British Journal of Psychology*, **32**, pp. 267–86.

Oldfield, R. C., and Zangwill, O. L. (1942b). Head’s concept of the schema and its application in contemporary British psychology. *British Journal of Psychology,* **33**, pp. 58–64.

Oldfield, R. C., and Zangwill, O. L. (1943). Head’s concept of the schema and its application in contemporary British psychology. *British Journal of Psychology*, **33**, pp.143–149.

O'Leary, Z. (2017). *The essential guide to doing your research project*. Sage.

Ollerton, M. (2006). *Getting the buggers to add up 2nd edition.* A and C Black.

O'Reilly, M., and Parker, N. (2013). 'Unsatisfactory Saturation': A critical exploration of the notion of saturated sample sizes in qualitative research. *Qualitative Research*, **13**(2), pp. 190-197.

Ormrod, J. E., and Cole, D. B. (1996). Teaching content knowledge and pedagogical content knowledge: A model from geographic education. *Journal of Teacher Education,* **47**(1), pp. 37-42.

Owens, D. T. (1980). Study of the relationship of area concept and learning concepts by children in grades three and four. In T. E. Kieren (Ed.), *Recent research on number concepts.* Columbus, Ohio: ERIC/SMEAC.

Pais, A. (2013*).* An ideology critique of the use-value of mathematics. *Educational Studies in Mathematics,* **84**(1), pp. 15-34.

Pais, A. (2017). The narcissism of mathematics education. In *The Disorder of Mathematics Education*. Springer, Cham. pp. 53-63

Pajares, M. F. (1992). Teachers’ beliefs and educational research: Cleaning up a messy construct. *Review of educational research*, **62**(3), pp. 307-332.

Pajares, M. F. (1996). Self-efficacy beliefs in academic settings. *Review of Educational Research,* **66**(4), pp. 543-578.

Pallascio, P., and Simmit, E. (2002). Philosophy for Children on Mathematics. In Elaine Simmt,. E and Davisd, A. (Eds.), *Proceedings of the ACTES Annual meeting*. Queen’s university. Ontario Canada.

Pang, J.S. (2000). Challenges in implementing reform ideals: Insights from South Korean and the U.S. elementary mathematics classrooms. *Paper presented at the Annual Meeting of the National Council of Teachers of Mathematics.* Chicago, IL.

Pang, J.S. (2001). Challenges of reform: Utility of sociomathematical norms. *Paper presented at the Annual Meeting of the American Educational Research Association.* Seattle, WA.

Pang, J.S. (2002). Difficulties and issues in applying the 7th mathematics curriculum to elementary school classrooms (in Korean). *School Mathematics*, **4**(4), pp. 657-675.

Pang, J.S. (2005). Transforming Korean Elementary Classrooms to student-centred Instruction. In Chick, H.L and Vincent, J.L (Eds.), *Proceedings of the 29th conference of the international group for the Psychology of mathematics Education,* **4**. Melbourne. Australia. pp. 41-48.

Parahoo, K. (2006). *Nursing research: principles, process and issues.* Palgrave Macmillan, Basingstoke.

Patton, M. Q. (2001). *Qualitative evaluation and research methods (3rd Ed.).* Thousand Oaks, CA: Sage Publications.

Pedrosa, A.M., Näslund, D. and Jasmand, C. (2012), Logistics case study based research: towards higher quality. *International Journal of Physical Distribution and Logistics Management*, **4**(23), pp. 275-295.

Perera, N. (2011). Constructivism, social constructivism and situated cognition: A sliding scale? [Web log post]. [Accessed 13 December 2018] <Available at : http://nishancperera. com/2011/01/31/constructivism-social-constructivismand-situated-cognition-a-sliding-scale-by-nishanperera>

Perez-Felkner, L., Nix, S., and Thomas, K. (2017). Gendered Pathways: How Mathematics Ability Beliefs Shape Secondary and Postsecondary Course and Degree Field Choices. *Front Psychology,* **8,** article 386.

Perry, B., Hampshire, A., Gervaxoni, A., and O'Neill, W. (2016). "Let's count": Improving community approaches to early years mathematics learning, teaching and dispositions through noticing, exploring and talking about mathematics. *Mathematics Education Research Group of Australasia.*

Perryman, J., Ball, S.J., Maguire, M., and Braun, A. (2011). Life in the pressure cooker—School league tables and English and mathematics teachers: Responses to accountability in a results-driven era. *British Journal of Educational Studies,* **59**(2), pp. 179–195.

Peterson, P. L., and Fennema, E. (1985). Effective teaching, student engagement in classroom activities, and sex-related differences in learning mathematics. *American Educational Research Journal*, **22**(3), pp. 309-335.

Piaget, J. (1936). *Origin of intelligence in the child.* London: Penguin books.

Piaget, J. (1952). *The origins of intelligence in the child (M. Cook, trans.).* New York: International Universities Press.

Piaget, J. (1959). *The language and thought of the child. (3rd Ed.).* London: Routledge.

Piaget, J. (1973). *The child's conception of the world.* London, Paladin.

Piaget, J. (1976). *'Piaget's Theory'*. In Piaget and His School. Springer, Berlin, Heidelberg, pp. 11-23.

Pimm, D. (1987). *Speaking Mathematically: Communication in Mathematics Classrooms.* Routledge. London.

Pimm, D. (1995). *Symbols and Meanings in School Mathematics.* London: Routledge.

Pink, S. (2007). *Doing visual ethnography (2nd Ed.).* London, United Kingdom: Sage.

Pinnegar, S., and Daynes, J. G. (2007). Locating narrative inquiry historically: Thematics in the turn to narrative. In D. J. Clandinin (Ed.), *Handbook of narrative inquiry: Mapping a methodology* pp. ix–xvii. Thousand Oaks: Sage Publications.

Pintrich, P. (2002). Future challenges and directions for theory. In B. Hofer and P. Pintrich (Eds.), *Personal epistemology: The psychological beliefs about knowledge and knowing.* Mahwah, NJ: Erlbaum. pp 389–414.

Polit, D. F., and Beck, C. T. (2004). *Nursing research: Principles and methods (7th Ed.).* Philadelphia PA.: Lippincott Williams and Wilkins.

Polly, D., McGee, J. R., Wang, C., Lambert, R. G., Pugalee, D. K., and Johnson, S. (2013). The association between teachers' beliefs, enacted practices, and student learning in mathematics. *Mathematics Educator,* **22**(2), pp. 11-30.

Polya, G. (1981). *Mathematical Discovery (Combined paperback edition).* New York: Wiley.

Post, T., Cramer, K., Behr, M., Lesh, R., and Harel, G. (1993). Curriculum implications of Research on the Learning, Teaching, and Assessing of Rational Number Concepts. In Carpenter, T. and Fennema, E. (Eds.), *Research on the Learning, Teaching, and Assessing of Rational Number Concepts.* Lawrence Erlbaum and Associates.

Powney, J., and Watts, M. (2018). *Interviewing in educational research*. Routledge.

Prasad, P. (2005). *Crafting qualitative research: Working in the postpositivist traditions.* New York: Armonk.

Prawat, R. S.,and Floden, R. E. (1994). Philosophical perspectives on constructivist views of learning. *Educational Psychologist*, **29**(1), pp. 37-48.

Pring, R. (2004) Philosophy of Educational Research. London: Continuum.

Putnam, R. T., and Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher*, **29**(1), pp. 4-15.

Putney, L. (2010). Case study. In N. Salkind (Ed.), *Encyclopaedia of research design.* Thousand Oaks, CA: SAGE Publications, pp.116-120.

QCA (2007). *Mathematics programme of study: Key stage 3.* London.

Queensland College of Teachers (QCT). (2012). *Australian professional standards for teachers.* [Accessed 1 September, 2013] <Available at: https://www.aitsl.edu.au/teach/standards>.

Radford, L. (2003). 'From truth to efficiency: comments on some aspects of the development of mathematics education'. *Canadian Journal of Science, Mathematics and Technology Education / Revue canadienne de l'enseignement des sciences, des mathématiques et des technologies,* **4**(4), pp. 551–556.

Radford, L., and Peirce, C. (2006). Algebraic thinking and the generalization of patterns: A semiotic perspective. *Paper presented at the Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education.* North American. pp. 2-21.

Ragin, C. C. (1992). Introduction: Cases of "What is a case?". *What is a case*, pp. 1-17.

Ramani, S., Könings, K. D., and Mann, K. (2018). A Guide to Reflexivity for Qualitative Researchers in Education. *Academic medicine: journal of the Association of American Medical Colleges*, **93**(8)**,** p. 1257.

Raphael, D., and Wahlstrom, M. (1989). The influence of instructional aids on mathematics achievement. *Journal for Research in Mathematics Education,* **20**(2), pp. 173–190.

Rasmussen, C., Zandieh, M., King, K., and Teppo, A. (2005). Advancing mathematical activity: A practice-oriented view of advanced mathematical thinking. *Mathematical Thinking and Learning,* **7**(1), pp. 51-73.

Ratcliff, D. (2003). Video methods in qualitative research. In P. M. Camic, J. E. Rhodes, and L. Yardley (Eds.), *Qualitative research in psychology: Expanding perspectives in methodology and design*. Washington, DC: American Psychological Association, pp. 113–129

Reason, P., and Bradbury, H. (2008). *Handbook of Action Research.* London. Sage.

Reinup, R. (2009). Developing of mathematics teachers' community: Five groups, five different ways. *Cerme 6–Working Group,* **10**, pp. 18-31.

Remillard, J. T. (1999).Curriculum materials in mathematics education reform: A framework for examining teachers' curriculum development. *Curriculum Inquiry*, **29**(3), pp. 315-342.

Resnick, L. B. (1990). Literacy in school and out. *Daedalus,* **119**(2), pp.169-185.

Resnick, L.B., and Ormanson, S. (1989). Developing thinking abilities in arithmetic class. *Paper presented to the Third European Conference for research on Learning and Instruction.* Madrid.

Rezat, S., Visnovska, J., Trouche, L., Qi, C., and Fan, L. (2018). Present Research on Mathematics Textbooks and Teachers' Resources in ICME-13: Conclusion and Perspectives. In: Fan L., Trouche L., Qi C., Rezat S., Visnovska J. (Eds.), *Research on Mathematics Textbooks and Teachers' Resources*. ICME-13 Monographs. Springer.

Rich, K. (2018). Building Conceptual Understandings of Equivalence.

Richardson, T. E. (2005). *Students' approaches to learning and teachers' approaches to teaching in higher education.* New York, NY: Routledge.

Richardson, V. (1997). Constructivist teaching and teacher education: Theory and practice. In *Constructivist Teacher Education: Building a World of New Understandings*, pp. 3-14.

Riggs, I. M., and Enochs, L. G. (1990). Toward the development of an elementary teacher's science teaching efficacy belief instrument. *Science Education*, **74**(6), pp. 625-637.

Ritchie, J. and Lewis, J. (2003). Generalising from qualitative research. Qualitative Research Practice. In *A Guide for Social Science Students and Researchers*. Sage.

Rivkin, S. G., Hanushek, E. A., and Kain, J. F. (2005). Teachers, schools, and academic achievement. *Econometrica*, **73**(2), pp. 417-458.

Roberts, N., and Foster, D. (2015). *Initial Teacher Training in England.* House of Commons Library Report, London, UK.

Robson, C. (2002). Real World Research 2nd Edition. Oxford: Blackwell

Rockoff, J. E., Jacob, B. A., Kane, T. J., and Staiger, D. O. (2011). Can you recognize an effective teacher when you recruit one? Education, 6(1), pp. 43-74.

Ross, J. A. (1992). Teacher efficacy and the effects of coaching on student achievement. *Canadian Journal of Education/Revue Canadienne De l'Education,* pp. 51-65.

Ross, J. A., McDougall, D., Hogaboam-Gray, A., and LeSage, A. (2003). A survey measuring elementary teachers' implementation of standards-based mathematics teaching. *Journal for Research in Mathematics Education*, pp. 344-363.

Rowe, D. E. (2018). Is (Was) Mathematics an Art or a Science? In *A Richer Picture of Mathematics.* Springer, Cham. pp. 407-411.

Rowan, B., Chiang, F., and Miller, R. J. (1997). Using research on employees' performance to study the effects of teachers on students' achievement. *Sociology of Education*, pp. 256-284.

Rowan, B., Correnti, R., and Miller, R. J. (2002). What large-scale, survey research tells us about teacher effects on student achievement: Insights from the Prospects study of Elementary Schools. *Teachers College Record,* **104**, pp. 1525-1567.

Ruffell, M, Mason, J. and Allen, B. (1998). *Studying attitude in mathematics. Educational Studies in Mathematics*, **35**, pp. 1-18.

Ruhleder, K., and Jordan, B. (1997). Capturing complex, distributed activities: Video-based interaction analysis as a component of workplace ethnography. In A. S. Lee, J. Liebenau and J. I. DeGross (Eds.), *Information systems and qualitative research.* London: Chapman and Hall. pp. 246-275.

Ruzic, R., and O'Connell, K. (2001). *Manipulatives.* Enhancement Literature Review, [Accessed 12 October 2015] <Available at : http://www.cast.org/ncac/Manipulatives1666.cfm>.

Sadler, A. (1984). *Pick and choose: A resource bank of questions for CSE / 'O' level*. Edward Arnold.

Salzman, P. C. (2002). On reflexivity. *American Anthropologist*, **104**(3), pp. 805-811.

Sambo, A, A. (1980). *Transfer effects of measure concepts on the learning of fractional numbers.* (Doctoral dissertation). University of Alberta, Canada.

Santagata, R., Zannoni, C., and Stigler, J. (2007). The role of lesson analysis in preservice teacher education: an empirical investigation of teacher learning from a virtual video-based field experience. *Journal of Mathematics Teacher Education*, **10**(2), pp. 123-140.

Santiago-Delefosse, M., Gavin, A., Bruchez, C., Roux, P., and Stephen, S. L. (2016). Quality of qualitative research in the health sciences: Analysis of the common criteria present in 58 assessment guidelines by expert users. *Social Science and Medicine*, **148**, pp. 142– 151.

Sarac, A., and Tutak, F. A. (2017). The relationship between teacher efficacy, and students' trigonometry self-efficacy and achievement. *International Journal for Mathematics Teaching and Learning*, **18**(1), pp. 66-83.

Sarama, J., and Clements, D. H. (2016). Physical and Virtual Manipulatives: What Is “Concrete”? In *International perspectives on teaching and learning mathematics with virtual manipulatives,* Springer, Cham, pp. 71-93.

Savin-Baden, M., and Major, C. H. (2013). Qualitative research: The essential guide to theory and practice. Abingdon, UK, Routledge.

Schank, R. C. (1975). *Conceptual information processing.* Amsterdam: North Holland.

Schifter, D., and Simon, M. A. (1992). Assessing teachers’ development of a constructivist view of mathematics learning. *Teaching and Teacher Education,* **8**, pp. 187–197.

Schmidt, C. (2004). The analysis of semi-structured interviews. *A companion to qualitative research*, pp. 253-258.

Schneider, R., and Pickett, M. (2006). Bridging engineering and science teaching: A collaborative effort to design instruction for college students. *School Science and Mathematics*, **106**(6), pp. 259-266.

Schoenfeld, A. H. (1985). *Mathematical Problem Solving.* New York: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: problem Solving, metacognition, and the sense of making mathematics. In Grouws, D. (Ed.), *Handbook for research on mathematics teaching and learning.* New York. MacMillian.

Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context*. Issues in Education*, **4**(1), pp. 1-94.

School Mathematics Project (SMP) (1961). *SMP Books One to Five, Books T and T4.* C.U.P., London.

Schrader, D. E. (2015). Constructivism and learning in the age of social media: Changing minds and learning communities. *New Directions for Teaching and Learning*, **144**, pp. 23-35.

Schuck, S., and Kearney, M. (2006). Using digital video as a research tool: Ethical issues for researchers. *Journal of Educational Multimedia and Hypermedia,* ***15***(4), pp. 447–463.

Schwandt, T.A. (2007). *Dictionary of qualitative inquiry. (3rd Ed.).* London: Sage.

Seale, C. (1999). Quality in qualitative research*. Qualitative Inquiry*, **5**(4), pp. 465-478.

Sewell, B., and Advisory Council for Adult and Continuing Education. (1981). *Use of mathematics by adults in daily life*. ACACE Leicester.

Sharma, M., Saxena, D., and Singh, V. (2017). *Unit-2 approaches and techniques of teaching mathematics.* [Accessed 12 June 2017] <Available at : http://14.139.40.199/bitstream/123456789/6689/1/Unit-2.pdf>.

Shaw, J. M. (2002). *Manipulatives enhance the learning of mathematics.* [Accessed 12 October 2015] <Available at : http://www.eduplace.com/state/author/shaw.pdf.>.

Shayer, M. (1999). Cognitive acceleration through science education II: Its effects and scope. *International Journal of Science Education*, **21**(8), pp. 883-902.

Shayer, M., and Adhami, M. (2003). Realising the cognitive potential of children 5–7 with a mathematics focus. *International Journal of Educational Research*, **39**(7), pp. 743-775.

Shayer, M., and Adhami, M. (2006). Fostering Cognitive Development through the context of Mathematics: Results of the CAME Project. *Educational Studies in Mathematics.*

Shayer, M., and Adhami, M. (2006a). The long-term effects from the use of CAME (Cognitive Acceleration in Mathematics Education), some effects from the use of the same principles in Y1 and 2, and the maths teaching of the future. *Research into Learning Mathematics*, **97**.

Shipman, M. D. (1997). *The limitations of social research (4th Ed.).* Harlow: Addison Wesley Longman.

Shuard, H. S. (1982). Differences in Mathematical Performance Between Girls and Boys. In Cockcroft, W. H. (Chair), *Mathematics Counts.* London: HMSO, pp. 273-287.

Shuard, H. S. (1986). The Relative Attainment of Girls and Boys in Mathematics in the Primary Years. In Burton, L. (Ed.), *Girls Into Maths Can Go.* London: Cassell, pp. 22-37.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, **15**(2), pp. 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, **57**(1), pp.1-23.

Shulman, L. S. (1999). Knowledge and teaching: foundations of the new reform. In J. Leach and B. Moon (Eds.), *Learners and pedagogy* (London, Paul Chapman), pp. 61–77.

Siegler, R. S., and Pyke, A. A. (2013). Developmental and Individual Differences in Understanding of Fractions. *Developmental Psychology*, **10**(49), pp. 1994-2004.

Siegler, R.S., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L. and Wray, J. (2010). Developing Effective Fractions Instruction for Kindergarten through 8th Grade. IES Practice Guide. NCEE 2010-4039. *What Works Clearinghouse*.

Sigurdson, S. E., and Olson, A. T. (1992). Teaching mathematics with meaning. *Journal of Mathematical Behavior,* **11**(1), pp. 37-57.

Sikes, P. (2004). *Methodology, procedures and ethical concerns. Doing Educational research a guide for first time researchers.* London, Sage.

Silver, E. A. (1994). On Mathematical Problem Posing Author*. For the Learning of Mathematics*, **14**(1), pp. 19-28.

Silver, E. A. (2017). Toward Understanding the Influence of Curriculum Resources on Students' Mathematics Learning: Cross-National Perspectives on What Matters Where, When, and for Whom. In Son JW., Watanabe T., Lo JJ. (Eds.), *What Matters? Research Trends in International Comparative Studies in Mathematics Education. Research in Mathematics* Education. Springer, Cham, pp.115-120.

Silverman, D. (2013). *Doing qualitative research: A practical handbook.* Sage Publications Limited.

Silverman, J., and Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education*, **11**(6), pp. 499-511.

Simon, M. (2013). Developing theory for design of mathematical task sequences: Conceptual learning as abstraction. *Paper presented at the ICMI Study 22: Task Design in Mathematics Education.* pp. 501-508.

Simons, H. (2009). *Case study research in practice*. SAGE publications.

Sjölund, S. (2018). These are the reasons we teach math: A study of teachers' cultural repertoire of discourses about the purposes of mathematics education.

Skemp, R. R. (1962). The need for a schematic learning theory*. British Journal of Educational Psychology,* **32**, pp. 133–142.

Skemp, R. R. (1971). *The Psychology of Learning Mathematics.* Middlesex, England: Penguin.

Skemp, R. R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher,* **26**(3), pp. 9–15.

Skemp, R. R. (1979). *Intelligence, Learning and Action.* Chichester: Wiley.

Skemp, R. R. (1986). *The Psychology of Learning Mathematics. (2nd Edition).* Middlesex, England: Penguin.

Skemp, R. R. (2006). *Relational understanding and instrumental understanding.* Mathematics Teaching in the Middle School, ***12***(2), 88–96. (Original work published in 1976).

SkillsYouNeed. (2017). *Soft skills.* [Accessed 1 June 2017] <Available at https://www.skillsyouneed.com/general/soft-skills.html>.

Skott, J. (2009). Contextualising the notion of 'belief enactment'. *Journal of Mathematics Teacher Education*, **12**(1), pp. 27-46.

Small, M. (2009). *Big ideas from dr. small grade 4-8: Creating a comfort zone for teaching mathematics kit.* Nelson.

Smith, A. (2004). *Making mathematics count: The report of Professor Adrian Smith's inquiry into post-14 mathematics education.* The Stationery Office.

Sowder, L. (1971). Performance on some discovery tasks: grades 4-7. *Journal for Research in Mathematics Education,* **2**, pp. 5-11.

Sowder, L., and Harel, G. (1998). Types of students' justifications. *The Mathematics Teacher*, **91**(8), pp. 670-675.

Sowell, E. J. (1989). 'Effects of manipulative materials in mathematics instruction'. *Journal for Research in Mathematics Education,* **20**(5), pp. 498–505.

Speer, N. M. (2005). Issues of Methods and Theory in the Study of Mathematics Teachers' Professed and Attributed Beliefs. *Educational Studies in mathematics,* **58**(3), pp. 361-391

Stacks, D., W. (2005). Case study. In R. Heath (Ed.), *Encyclopaedia of Public Relations.* Thousand Oaks, CA: SAGE Publications, pp. 118-119.

Stake, R. E. (1995). *The art of case study research.* London: Sage.

Stake, R. E. (2008) Qualitative Case Studies. In N. K. Denzin and Y. S. Lincoln (Eds.), *Strategies of Qualitative Inquiry*. Sage Publications: Los Angeles. Chapter. 17.

Stankov, L., and Lee, J. (2008). Confidence and cognitive test performance. *Journal of Educational Psychology,* **100**(4), pp. 961-976.

Stankov, L., Lee, J., Luo, W., and Hogan, D. J. (2012). Confidence: A better predictor of academic achievement than self-efficacy, self-concept and anxiety? *Learning and Individual Differences*, **22**(6), pp. 747-758.

Starman, J., Larson, A., Proffitt, E., Guskey, T., and Ma, X. (2013). A collaborative professional development approach to improving student outcomes. *Kentucky Journal of Excellence in College Teaching and Learning*, **11**(1), pp. 81 – 86

Steadman, S., Eraut, M., Fielding, M., and Horton, A. (1995). *Making school-based INSET effective.* University of Sussex, Institute of Education Research Reports.

Steadman, S., Eraut, M., Fielding, M., and Horton, A. (1995). Making school-based INSET effective. Research report 2*. University of Sussex, Institute of Education Research Reports.*

Steffe, L. P. (1983)*.* Children's algorithms as schemes. *Educational Studies in Mathematics,* **14**, pp. 109–125.

Steffe, L. P. (1988). Children's construction of number sequences and mutliplying schemes. In Hiebert, J. and Behr, M. (Eds*.), Number Concepts and Operations in the Middle Grades.* Reston, VA: The National Council of Teachers of Mathematics, **2**, pp. 119–140.

Steffe, L. P. (1990*). Overview of the action group al: Early childhood years. Transforming Children's Mathematics Education: International Perspectives,* pp. 3-15.

Steffe, L. P., and Gale, J. (Ed.). (1995). *Constructivism in Education.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Steffe, L. P., Thompson, P. W., and Richards, J. (1982). Children's counting in arithmetical problem solving. *Addition and Subtraction: A Cognitive Perspective*, pp. 83-97.

Stein, M. K., and Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation*, **2**, pp. 50-80.

Stenbacka, C. (2001*).* Qualitative research requires quality concepts of its own. *Management Decision,* **39**(7), pp. 551-556.

Stevenson, M. (2010). Conceptions of 'Understanding mathematics in depth': What do teachers need to know and how do they need to know it? Informal *Proceedings of the British Society for Research into Learning Mathematics,* **30**(3), pp. 79-84.

Stewart, A. (2014). Case study. In Mills, J. and Birks, M. (Eds.), *Qualitative methodology: A practical guide*. Thousand Oaks, CA: Sage. pp.145-159.

Stigler, J. W., and Hiebert, J. (2009). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom* (First Free Press trade paperback edition). New York: Free Press.

Stigler, J. W., Fernandez, C., and Yoshida, M. (1996). Cultures of mathematics instruction in Japanese and American elementary classrooms. In Rohlen, T. P. and LeTendre, G.K. (Eds.), *Teaching and learning in Japan.* Cambridge: Cambridge University Press. pp. 213–247.

Stitt-Gohdes, W. L. (2001). *Business education students' preferred learning styles. Journal of Career and Technical Education*, **18**(1). [Accessed 1 May 2015]< Available at :http://scholar.lib.vt.edu/ejournals/JCTE/v18n1/stitt.html>.

Stobart, G., and Stoll, L. (2005). *The key stage 3 strategy: What kind of reform is this? Cambridge Journal of Education*, **35**(2), pp. 225-238.

Strauss, A., and Corbin, J. (1990). *Basics of grounded theory methods.* Beverly Hills: Sage.

Suydam, M.N. (1986). ‘Manipulative materials and achievement’. *Arithmetic Teacher,* **33**(6), pp. 10 -32.

Swan, M (2005). *Improving Learning in Mathematics: challenges and strategies*. DfES Standards Unit Publication.

Swan, M. (2006). *Collaborative Learning in Mathematics: A Challenge to our Beliefs and Practices.* London: National Institute for Advanced and Continuing Education (NIACE.for the National Research and Development Centre for Adult Literacy and Numeracy (NRDC).

Swann Report (1985) Education for All. HMSO. London.

Taleporos, E. (2005). Consequential validity: A practitioner’s perspective. *Educational Measurement: Issues and Practice*, **17**(2), pp. 20–23.

Tall, D. (2002). Continuities and Discontinuities in Long-Term Learning Schemas. In Tall, D. and Thomas, M. (Eds.), *Intelligence, Learning and Understanding - A Tribute to Richard Skemp.* Brisbane, Australia: Post Pressed. pp. 151-177.

Tannen, D. (1993). Introduction. In Tannen, D. (Ed.), *Framing in discourse* New York: Oxford University Press. pp. 3-13.

Tatto, M. T., Peck, R., Schwille, J., Bankov, K., Senk, S. L., Rodriguez, M., Ingvarson, L., Reckase, M., and Rowley, G. (2012). *Policy, Practice, and Readiness to Teach Primary and Secondary Mathematics in 17 Countries: Findings from the IEA Teacher Education and Development Study in Mathematics (TEDS-MM)*. International Association for the Evaluation of Educational Achievement. Herengracht 487, Amsterdam, 1017 BT, The Netherlands.

Tereshchenko, A., Francis, B., Archer, L., Hodgen, J., Mazenod, A., Taylor, B., Pepper, D., and Travers, M. (2018). *Learners’ attitudes to mixed attainment grouping: examining the views of students of high, middle and low attainment.* Research Papers in Education, DOI: 10.1080/02671522.2018.1452962.

Tevebaugh, T. N. (1998). Mathematics is not a universal language. *Teaching Children Mathematics,* **5**(4), pp. 214-216.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research*.* In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning.* New York: Macmillan. pp. 127-146.

Thompson, A. C. (2012). The effect of enhanced visualization instruction on first grade students' scores on the North Carolina standard course assessment (Dissertation, Liberty University, Lynchburg, VA).

Thompson, A. G. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics*, **15**(2), pp. 105-127.

Thompson, D., and Rubenstein, R. (2000). Learning mathematics vocabulary: Potential pitfalls and instructional strategies. *Mathematics Teacher*, **93**(7), pp. 568-574.

Thompson, I. (2003). *Enhancing Primary Mathematics Teaching*. McGraw-Hill. UK.

Thompson, P. W., and Saldanha, L. A. (2003). Fractions and multiplicative reasoning. *Research Companion to the Principles and Standards for School Mathematics,* pp. 95-113.

Thompson, P.W., and Thompson, A.G. (1984). Talking about Rates conceptually, part1: A teachers' struggle. *Journal of Research in Mathematics Education*, **25**, pp. 279-303.

Thomson, S. B. (2011). Qualitative research: Validity. *JOAAG*, **6**(1), pp. 77-82.

Torrance, H. (2017*).* Blaming the victim: assessment, examinations, and the responsibilisation of students and teachers in neo-liberal governance. *Discourse: Studies in the Cultural Politics of Education,* **38**(1), pp. 83-96

Tosto, M. G., Asbury, K., Mazzocco, M. M., Petrill, S. A., and Kovas, Y. (2016). From classroom environment to mathematics achievement: The mediating role of self-perceived ability and subject interest. Learning and individual differences, **50**, pp. 260-269.

Tournaki, N. (2003). *The differential effects of teaching addition through strategy instruction versus drill and practice to students with and without learning disabilities. Journal of Learning Disabilities*, **36**, pp. 449–458.

Tournaki, N., and Podell, D. M. (2005). *The impact of student characteristics and teacher efficacy on teachers' predictions of student success. Teaching and Teacher Education,* **21**(3), pp. 299-314.

Tschannen-Moran, M., Hoy, A. W., and Hoy, W. K. (1998*).* Teacher efficacy: Its meaning and measure. *Review of Educational Research,* **68**(2), pp. 202-248.

Tuohy, D., Cooney, A., Dowling, M., Murphy, K., and Sixsmith, J. (2013). An overview of interpretive phenomenology as a research methodology. *Nurse Researcher*, **20**(6), pp. 17-20.

Twining, P., Heller, R., Nussbaum, M., and Tsai, C.-C. (2017). Some guidance on conducting and reporting qualitative studies. *Computers and Education*, **106**(1), A1-A9.

Ulewicz, M., and Beatty, A. (Eds.) (2001). *The Power of Video Technology in International Comparative Research in Education*. Washington D.C.: National Academy Press.

UNESCO (2005). *UN Millennium Summit delivers rhetoric without commitment*, Paris: UNESCO, Global Campaign for Education.

United States Department Education (USDE) (2008). *Foundations for Success. The Final Report of the National Mathematics Advisory Panel.* [Accessed 9 January 2012] <Available at : https://www2.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf>.

Usher, R., and Edwards, R. (1994). *Postmodernism and education: Different voices, different worlds.* London: Routledge.

Usher, R., and Solomon, N. (1999*).* Experiential learning and the shaping of subjectivity in the workplace. *Studies in the Education of Adults*, **31**(2), pp. 155- 163.

Usiskin, Z. (1996*).* Mathematics as a language. In *Communication in Mathematics K-12 and Beyond: 1996 Yearbook.* Reston VA: National Council of Teachers of Mathematics. pp. 231-243.

Van Maanen, J. (1979). Reclaiming qualitative methods for organizational research: A preface. *Administrative Science Quarterly,* **24**(4), pp. 520-526.

Van Maanen, J., Dabbs, J. M., and Faulkner, R. R. (1982). *Varieties of qualitative research*. Sage Publications, Inc.

Vanblaere, B., and Devos, G. (2018). The Role of Departmental Leadership for Professional Learning Communities. *Educational Administration Quarterly,* **54**(1), pp. 85-114.

Veel, R. (1999). Language, knowledge and authority in school mathematics. In F. Christie (Ed.), *Pedagogy and the shaping of consciousness: Linguistic and social processes.* New York: Continuum. pp. 185–216.

Veenman, S., Lem, P., and Roelofs, E. (1989). Training teachers in mixed age classrooms: effects of a staff development programme. *Educational Studies*, **15**(2), 165-180.

Vergnaud, G. (1983). Multiplicative structures. In Lesh, R. and Landau, M. (Eds.), *Acquisition of mathematics concepts and processes.* London: Academic Press. pp. 127-174.

Verma, K.G., and Mallick, K. (1999). *Researching education: Perspectives and techniques.* Philadelphia: Falmer Press.

Vieira, L., and Auriemma, D. (2015). Effective Communication Needs Common Language and Goals. *Communication Skills for Leaders,* **15**(10), no .page.

vom Lehn, D., and Heath, C. (2007). Social interaction in museums and galleries: A note on video-based field studies. In R. Goldman, R. D. Pea, B. Barron and S. J. Derry (Eds.), *Video research in the learning sciences.* Mahwah, NJ: Lawrence Erlbaum Associates. pp. 287-301.

Vorderman, C., Budd, C., Dunne, R., Hart, M., and Porkess, R. (2011). *A world class mathematics education for all our young people.* London: DfES.

Vulliamy, G., and Webb, R. (1995). The implementation of the national curriculum in small primary schools*. Educational Review*, **47**(1), pp. 25-41.

Vygotsky L. S. (1978*). Mind in Society. The Development of Higher Psychological Processes.* Harvard University Press.

Vygotsky, L. S. (1980). *Mind in society: The development of higher psychological processes.* Harvard University Press.

Walker, E.N. (2016). The Importance of Communities for Mathematics Learning and Socialization. *Journal of Urban Mathematics Education*, **9**(2), pp. 5–10.

Watson, A., and Mason, J. (2006). Seeing an Exercise as a Single Mathematical Object: Using Variation to Structure Sense-Making. *Mathematical Thinking and Learning,* **8**(2), pp. 91-111.

Watson, A., and Mason, J. (2007). Taken-as-shared: A review of common assumptions about mathematical tasks in teacher education. *Journal of Mathematics Teacher Education,* **10**(4), pp. 205-215.

Watson, A., and Ohtani, M. (2012). *Task design in mathematics education discussion document.* [Accessed 8 February 2013] <available at: www.mathunion.org/icmi/conferences/icmi-studies/ongoingstudies/> .

Watson, A., Drijvers, P., Senuma, H., Goldenberg, P., and Bloch, I. N. (2008). *Background statement, topic study group 34: Research and development in task design and analysis.* 11th International Congress on Mathematical Education. Monterrey, Mexico.

Watson, A., Jones, K. and Pratt, D. (2013). *Key Ideas in Teaching Mathematics: Research-based guidance for ages 9-19*. OUP Oxford.

Watson, S. (2019). Revisiting teacher decision making in the mathematics classroom: a multidisciplinary approach.

Wayne, A. J., and Youngs, P. (2003). Teacher characteristics and student achievement gains: A review. *Review of Educational Research,* **73**(1), pp. 89-122.

Webb, M., and Cox, M. (2004). A review of pedagogy related to information and communications technology. *Technology, Pedagogy and Education,* **13**(3), pp. 235–286.

Weegar, M. A., and Pacis, D. (2012). *A comparison of two theories of learning--behaviorism and constructivism as applied to face-to-face and online learning.* Paper presented at the Proceedings E-Leader a Conference, Manila. [Accessed 25th April 2015] Available at: < http://www.g-casa.com/conferences/manila/papers/Weegar.pdf>.

Weinstock, M., Kienhues, D., Feucht, F. C., and Ryan, M. (2017). Informed reflexivity: Enacting epistemic virtue. Educational Psychologist, **52(**4), pp. 284-298.

Weissglass, J. (1991). Teachers have feelings: What can we do about it? *Journal of Staff Development,* **12**(1), pp. 28-33.

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity.* Cambridge: Cambridge university press.

Wenger, E., McDermott, R., and Snyder, W. (2002*). Cultivating Communities of Practice: A Guide to Managing Knowledge.* Boston, MA: Harvard Business School Press.

Whiteford, T. (2009). Is mathematics a universal language? *Teaching Children Mathematics,* **16**(5), pp. 276-283.

Wiggins, G. P., and McTighe, J. (2011). *The understanding by design guide to creating high-quality units.* ASCD. Alexandra. VA. USA.

Williams, C. (2007). Research methods*. Journal of Business and Economics Research (JBER)*, **5**(3), pp. 65-71.

Williams, M., and Burden, R. (1997). *Psychology for language teachers: A social constructivist approach.* Cambridge: Cambridge University Press.

Willis, J. (1996). A flexible framework for task-based learning. *Challenge and Change in Language Teaching*, pp. 52-62.

Wilshaw M. (2012). *'High expectation, no excuses', speech at London Leadership Strategy*, [Accessed at : 27 April 2013] <Available at : http://www.ofsted.gov.uk/ news/high-expectation-no-excuses-sir-michael-wilshaw-hmcioutlines-changes-ofsted-inspection-drive-delive-0>.

Wilson, A. (2014). Being a practitioner: An application of Heidegger's phenomenology. *Nurse Researcher*, **21**(6), pp. 28-33.

Wilson, B. (2017). Constructivism for Active, Authentic Learning. *Trends and Issues in Instructional Design and Technology.* pp. 61-67.

Wilson, M., and Cooney T. (1992). Mathematics teacher change and development. the role of beliefs. In Leder, G.C., Pehkonen, E. and Torner G.

(Eds.), Beliefs: *A hidden variable in mathematics education?* Dordrecht, Netherlands: Kluwer Academic Publishers. pp. 127-146.

Wilson, S. M., Rozelle, J. J., and Mikeska, J. N. (2011). Cacophony or embarrassment of riches: Building a system of support for quality teaching. *Journal of Teacher Education,* **62**(4), pp. 383-394.

Witcher, L. A., Onwuegbuzie, A. J., Collins, K. M., Witcher, A. E., Minor, L. C., and James, T. L. (2002). Relationship between teacher efficacy and beliefs about education among preservice teachers. *Paper presented at the Annual Meeting of the Mid-South Educational Research Association.* Chattanooga, TN, 6- 8 November.

Witz, A. (1992). *Professions and patriarchy.* London: Routledge.

Wolfe, C. (2010). *What Is posthumanism*? Minneapolis: University of Minnesota Press.

Wolff, S. (2004). Analysis of documents and records. In Flick, U von Kardoff, E. and Steinke, I. (Eds.), (Translated by Jenner, B.). *A companion to Qualitative Research* (Sage, London), pp. 284-288.

Wong, K. Y. (2015). *Use of Student Mathematics Questioning To Promote Active Learning and Metacognition,* In *Selected Regular Lectures from the12th International Congress on Mathematics Education* London: Springer International Publishing, pp. 877–895.

Wood, T., Cobb, P. and Yackel, E. (1990). The contextual nature of teaching: Mathematics and reading instruction in one second-grade classroom. *The Elementary School Journal*, **90**(5), pp. 497-513.

Woodward, J. (2017). The concept of magnitude and what it tells us about how struggling students learn fractions. *Journal of learning disabilities*, **50**(6), pp. 640-643.

Woolfolk, A. E., Rosoff, B., and Hoy, W. K. (1990). Teachers' sense of efficacy and their beliefs about managing students. *Teaching and Teacher Education*, **6**(2), pp. 137-148.

Wright, D. L. (1988). Research on teaching mathematics: Making subject matter knowledge part of the equation. *Advances in research on teaching (2 teachers' subject matter knowledge and classroom instruction)*. Greenwich CT: Jai Press.

Wright, L.W. (1995*).* Qualitative International Management Research. In Punnett, B.J. and Shenkar, O. (Eds.), *Handbook for International Management Research*. Backwell, Oxford.

Yackel, E. (2002). ‘What we can learn from analyzing the teacher’s role in collective argumentation’. *Journal of Mathematical Behavior,* **21**, pp. 423 – 40.

Yackel, E., and Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education,* **27**(4), pp. 458-477.

Yates, S. M. (2006). Elementary teachers' mathematics beliefs and teaching practices after a curriculum reform. *Paper presented at the Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*, **5**, pp. 433-440.

Yin, R. K. (2013). Case study research: Design and methods (applied social research methods). *London and Singapore: Sage*.

Zhang, Q., and Stephens, M. (2013). Utilising a construct of teacher capacity to examine national curriculum reform in mathematics. *Mathematics Education Research Journal*, **25**(3), pp. 481 – 502.

Zwahlen, E. K. (2014). *An investigation of how preservice teachers design mathematical tasks.* All Theses and Dissertations. 3959. [Accessed 10 January 2017] <Available at: https://scholarsarchive.byu.edu/etd/3959

Appendix 1 – Survey questionnaire

Research into the pedagogical beliefs of mathematics teachers, PGCE students and their mentors.

Initial Information

Thank you for taking part in this research. The information you give will only be used in the context of my research and your privacy will be respected.

Name:	
Date:	

The information given on this form will only be used for research purposes and your privacy respected.

Please indicate the number of years you have been teaching.

Trainee / NQT	1-5 years	6-10 years	11-15 years	16-20 years	21+ years

Please indicate your gender

Male	Female

Please indicate which of the following applies

Trainee / NQT	Class teacher	Head of Dept.	Senior Leader

Please indicate which age range applies to you

21-30 yrs old	31-40 yrs old	41-50 yrs old	51-60 yrs old	60+ yrs old

Please indicate the percentage of the week you spend teaching mathematics

0-20%	21-40%	41-60%	61-80%	81-100%

Please indicate where you did your teacher training (PGCE).

Please indicate the title of your degree.

Approximately what percentage of your degree was mathematics?

0-20%	21-40%	41-60%	61-80%	81-100%

Please grade / rate the following statements using **your currently** held beliefs.

Please use the scale to rate the statements

I **almost never** do this when teaching mathematics.
I **occasionally** do this when teaching mathematics
I do this **about half of the time** when teaching mathematics
I do this **most of the time** when teaching mathematics
I do this **almost always** when teaching mathematics.

Please place a tick in the box which best represents what you currently think about the teaching of mathematics.

NB. For the purpose of this research
Skills is taken to mean pupils answering questions such
Expand 3(x+6)
or Factorise $x^2 - 5x + 6$

Tasks is taken to mean extended open- ended investigations such as
How many triangles can you draw on a 3 by 3 square isometric grid

or

On a square isometric grid draw any polygon. Count the dots on the perimeter and those inside the polygon. Can you find a connection
(Pick's Theorem)

	Almost never	Occasionally	About half of the time	Most of the time	Almost always
I think learners should spend time in every lesson practising mathematics skills.					
I think learners gain mathematical insight from practising skills.					
I think learners should mainly work on their own when practising skills.					
I think learners should tackle tasks.					
I think good mathematical tasks are difficult to design.					
I think good mathematical tasks have unforeseen learning outcomes.					
I think good mathematical tasks can lead learners along unproductive pathways.					
I think mathematical tasks can lead learners to incorrect mathematical conclusions.					
I think tasks are coursework in disguise.					
I think planning mathematical tasks is time consuming.					
I think mathematical tasks take up too much teaching time.					
I think mathematical tasks motivate learners.					
I think designing mathematical tasks is a complex exercise.					
I think designing mathematical tasks is not necessary as other resources are readily available.					
I think mathematical tasks do not easily fit the current lesson structure					

	Almost never	Occasionally	About half of the time	Most of the time	Almost always
I think learners feel the 3 part lesson supports their learning.					
I think learners feel confident and secure if I use one or two teaching approaches.					
I think learners feel confident and secure if I only use one or two teaching approaches.					
I think learners should be informed as to the learning objective when tackling tasks.					
I think learners should learn mathematics through discussing their ideas.					
I think learning dialogue (about mathematics) is important.					
I think learners should compare and share their solutions.					
I think learners should mainly work in pairs or groups.					
I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.					
I think it is easy to differentiate tasks for pupils					
I think differentiating tasks for pupils is a good idea.					

Thank you for your time and thoughts.
The information will only be used as part of my research.
If you would like to be informed about the findings of this research please contact me in July 2015.

Mike Rickhuss
Pathway Leader
Secondary PGCE Mathematics
University of Wolverhampton
Mike.Rickhuss@wlv.ac.uk

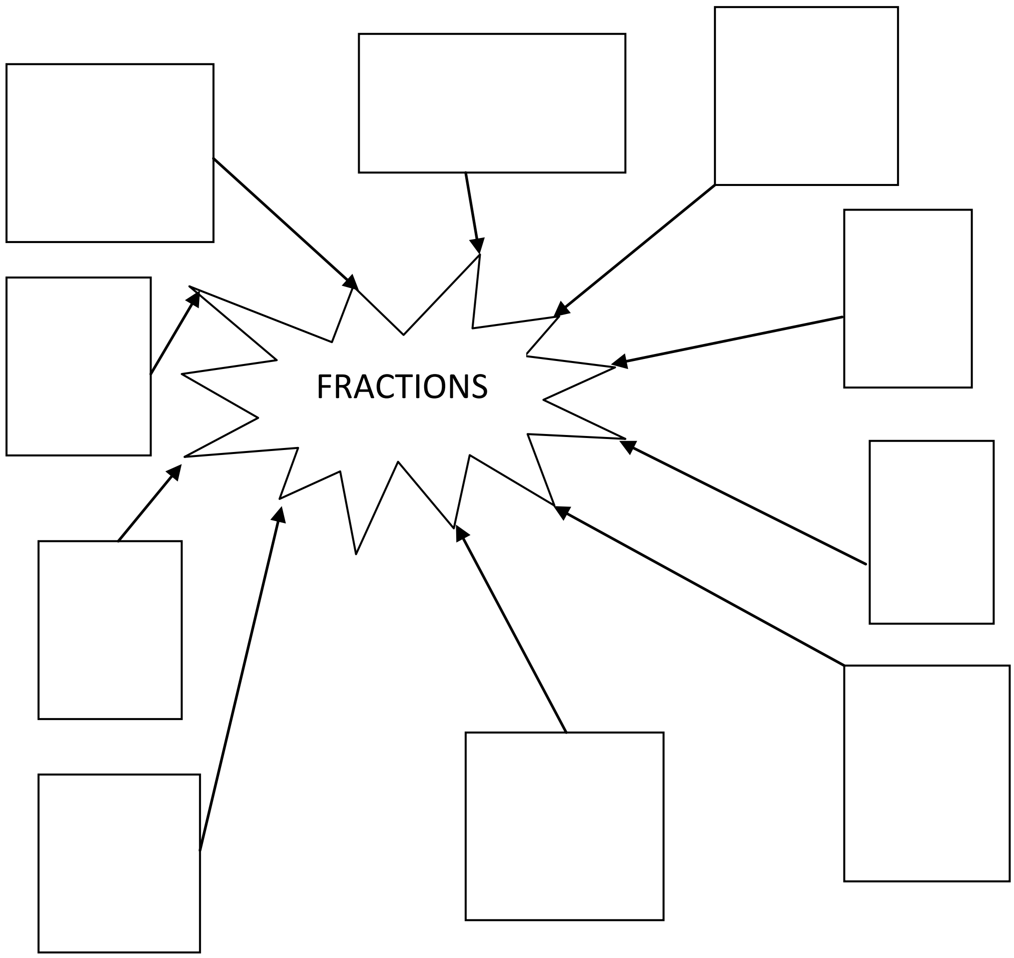
Appendix 2 – Questionnaire Categories (Coding)

No.	Question	Code
1	I think learners should spend time in every lesson practising mathematics skills.	S
2	I think learners gain mathematical insight from practising skills.	S
3	I think learners should mainly work on their own when practising skills.	S
4	I think learners should tackle tasks.	T
5	I think good mathematical tasks are difficult to design.	T
6	I think good mathematical tasks have unforeseen learning outcomes.	T
7	I think good mathematical tasks can lead learners along unproductive pathways.	T
8	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	T
9	I think tasks are coursework in disguise.	T
10	I think planning mathematical tasks is time consuming.	T
11	I think mathematical tasks take up too much teaching time.	T
12	I think mathematical tasks motivate learners.	T
13	I think designing mathematical tasks is a complex exercise.	T
14	I think designing mathematical tasks is not necessary as other resources are readily available.	T
15	I think mathematical tasks do not easily fit the current lesson structure.	T
16	I think learners feel the 3 part lesson supports their learning.	P
17	I think learners feel confident and secure if I use one or two teaching approaches.	P
18	I think learners feel confident and secure if I only use one or two teaching approaches.	P
19	I think learners should be informed as to the learning objective when tackling tasks.	P
20	I think learners should be informed as to the learning objective when tackling tasks.	P
21	I think learning dialogue (about mathematics) is important.	P
22	I think learners should compare and share their solutions.	P
23	I think learners should mainly work in pairs or groups.	P
24	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	S
25	I think it is easy to differentiate tasks for pupils	T
26	I think differentiating tasks for pupils is a good idea.	T

S = Skills, T = Task and P = Pedagogy

Appendix 3 – Study School Lesson Activity.

What do we know about fractions?



Appendix 4 - Study School Lesson - Skill

Names:........................

Fraction – Division – Worksheet 2a

HOW MANY

1. How many $\frac{1}{6}$' s are there in $\frac{2}{3}$?

2. How many $\frac{1}{8}$' s are there in $\frac{2}{4}$?

3. How many $\frac{1}{12}$' s are there in $\frac{2}{3}$?

4. How many $\frac{1}{16}$' s are there in $\frac{2}{8}$?

5. How many $\frac{1}{32}$' s are there in $\frac{2}{8}$?

This was done by : ……………………………………………………………………………………

Discussion – With a partner

With your partner – can you find a connection between the answer and the numbers in the fractions?

Write your answer to share with the class

We noticed that

____

____

____

____

Names:.........................

Fraction – Division – Worksheet 2b

Writing the questions mathematically

We know the answer to; how many $\frac{1}{6}'$ s are there in $\frac{1}{3}$? was 2.

So we could write $\frac{1}{3} \div \frac{1}{6} = 2$

Rewrite the questions for

1. How many $\frac{1}{8}'$ s are there in $\frac{1}{4}$?

So we could write

2. How many $\frac{1}{12}'$ s are there in $\frac{1}{3}$?

So we could write

3. How many $\frac{1}{8}$'s are there in $\frac{1}{2}$?

So we could write

4. How many $\frac{1}{32}$'s are there in $\frac{1}{8}$?

So we could write

Appendix 5 - Study School Lesson - Exercise

Names:.........................

Fraction – Worksheet

Which Fraction?

$\frac{1}{2}$ = $\frac{1}{4}$ + $\frac{1}{4}$

You do not need to use colours you can just write the fractions.

☐ = ☐ + ☐

☐ = ☐ + ☐

☐ = ☐ + ☐

☐ = ☐ + ☐

☐ = ☐ + ☐

☐ = ☐ + ☐

☐ = ☐ + ☐

☐ = ☐ + ☐

Names:..............................

Fraction – Division – Worksheet 3 - Context

The bottle has $1\frac{3}{4}$ *litres* of concentrated squash. In each glass the party host wants just $\frac{1}{8}$ *litre* of the concentrate before topping the glass up with water.

The problem is the host needs to find out how many glasses of squash she can make from one bottle of the concentrate. Can you help?

Explain (with diagrams if you wish) how you have found the answer

WHAT IF QUESTION

What if the bottle held $2\frac{1}{2}\,litres$ of concentrate and the glasses need $\frac{1}{12}\,litre$. How many glasses could the host fill from a bottle of the concentrate?

Explain (with diagrams if you wish) how you have found the answer

____

____

____

Discussion – With a partner Make up a "What if question" and find the

Our what if Question

....

....

Answer is ....

Fraction – Division – Worksheet 3 - Context

The bottle has $1\frac{3}{4}$ *litres* of concentrated squash. In each glass the party host wants just $\frac{1}{8}$ *litre* of the concentrate before topping the glass up with water.

The problem is the host needs to some to find out how may glasses of squash she can make from one bottle of the concentrate. Can you help?

What if Questions

What if the bottle held $2\frac{1}{2}$ *litres* of concentrate and the glasses need $\frac{1}{12}$ *litre* .

How many glasses could the host fill from a bottle of the concentrate?

Discussion – With a partner

Make up a "What if question" and find the answer.

Appendix 7 – Study lesson plan

Fraction – Division

Division of fractions is explored from a beginning of equivalent fractions and the concept of what division means.

Thinking strands
Equivalent fractions, division
Curriculum Links

Resources
Tarsia puzzles , worksheets

Lesson Summary

Learning Episode 1 (15 mins)

Go over the idea of equivalent fractions – use visual stimuli if required otherwise a open cloud **activity** to recall prior learning.

Teacher input (10 mins)

How did you know $\frac{4}{6} = \frac{8}{12}$
Explain how we find fractions equivalent to $\frac{3}{7}$? or $\frac{5}{11}$?
The class / group discussion should focus on the operation of multiplication and the multiplier is used to change the numerator and denominator.

Learning Episode 2 (10 mins)

Dividing two fractions. This is not approached by an algorithm. Look a division as how many. This is a **skill** to develop. How do we divide 1/6 by 1/3? Emphasise **how many**
$\frac{1'}{6}$ s arethere in $\frac{2}{3}$?

Use the tiles to demonstrate.
Only present a sheet with 5 questions and then allow the pupils to make up their own questions to solve.
Worksheet required.

Learning Episode 3 (15 mins)

Using the idea of equivalent fractions and adding – how could we solve **exercises** like

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$$

Only present a sheet with empty boxes to allow the pupils to make up their own questions to solve.

Worksheet required.

Learning Episode 4 (15 mins)

Real – life **task** problem about squash. With what if questions

Worksheet required.

Discussion (10 mins)

How can we divide two fractions when the denominators have a common factor?

Ex. $\frac{3}{5} \div \frac{3}{15}$?

How can we divide two fractions when the denominators are prime numbers?

Ex. . $\frac{3}{5} \div \frac{3}{7}$?

Mathematical Content

The activities focus on the skill of division as “How many of these are there in ...”
It takes the notions that cognitive schemas for division and equivalent fractions; once learnt and internalised; can and should be applied to the division of fractions.

The process has three distinct categories

a) fractions with common denominators

$$\frac{na}{b} \div \frac{a}{b}$$

b) fractions where denominators can be made equivalent where one is a factor of the other.

$$\frac{a}{b} \div \frac{c}{d} \quad \text{where nb = d}$$

$$\frac{na}{nb} \div \frac{c}{d}$$

c) fractions where denominators are not factors of each other (such as primes)

$$\frac{a}{p_1} \div \frac{b}{p_2}$$

where p_1 and p_2 are primes

Each of the three categories can produce two outcomes where the numerators divide and produce an integer or where the result in a fraction. The important concept is that once the denominators have been ‘made’ equivalent the resulting denominator division is unity.

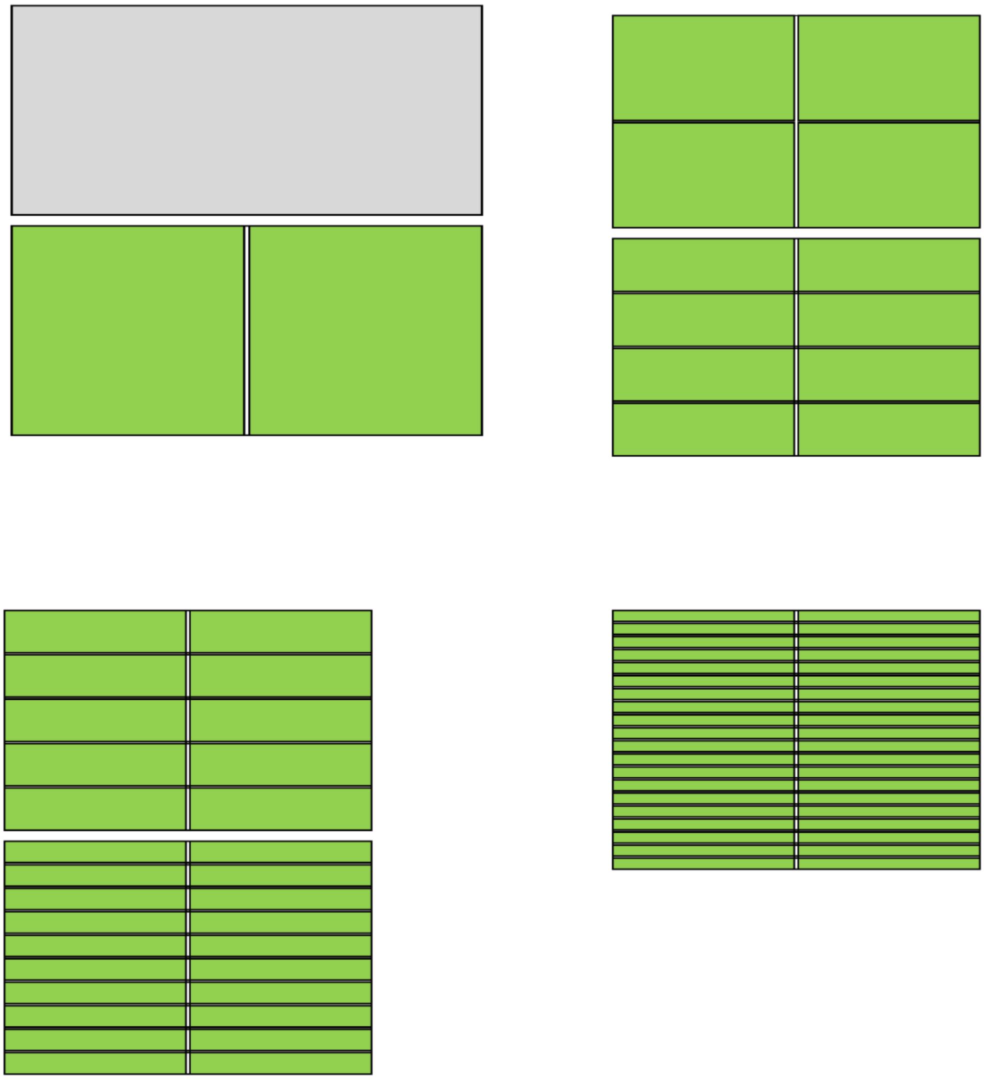
$$\frac{na}{b} \div \frac{a}{b}$$

Pupil Thinking

Pupils are encouraged to make the connection between division for integers and division for fractions. The discussion in activity 2 should centre around multiplication being the inverse operator for division and that this can be used to help with dividing.

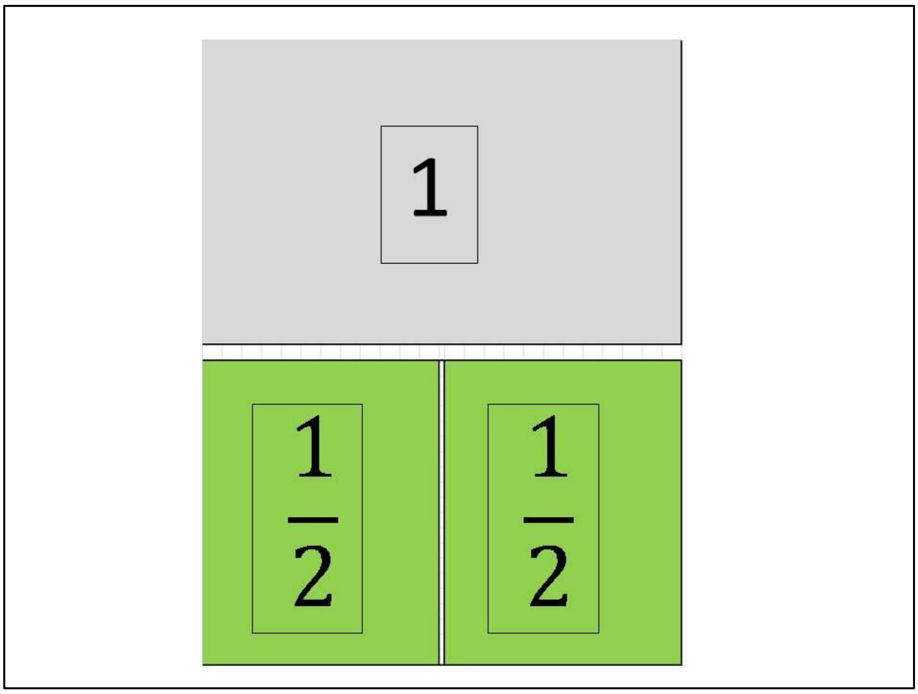
Appendix 8 - Study Lesson Manipulatives

An example of one set of Fraction manipulative tiles used in the study (based on a whole, half, quarter, eighth and sixteenth, a thirty second and sixty-fourth).

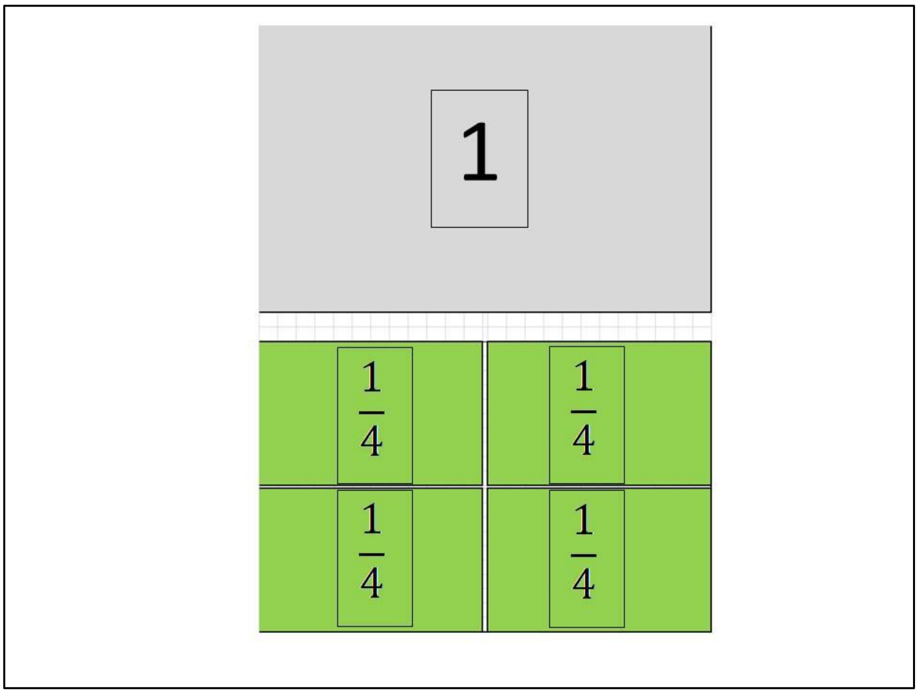


Appendix 9 - Study lesson PowerPoint slides

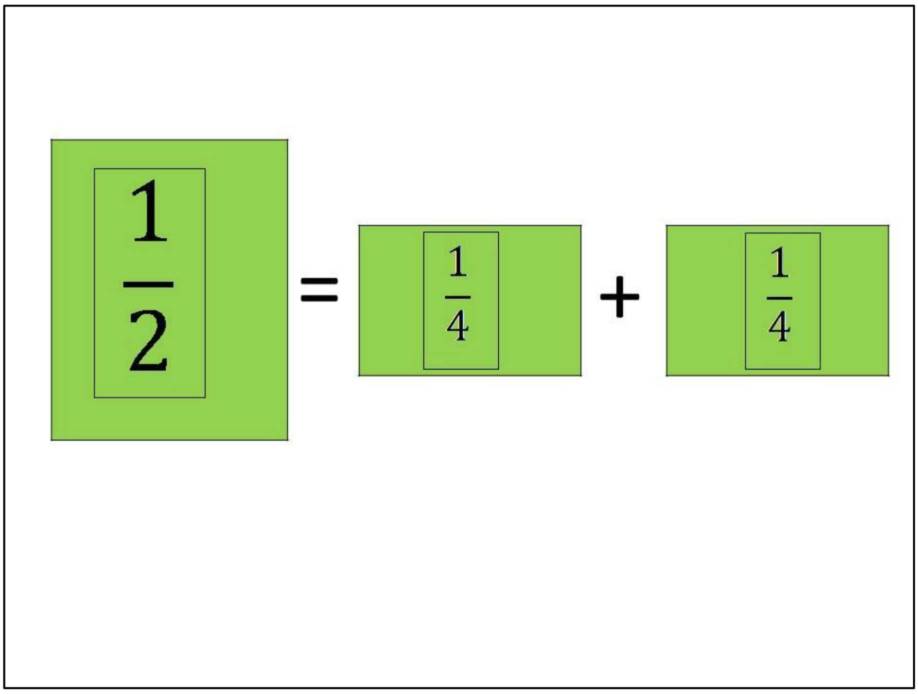
Slide1



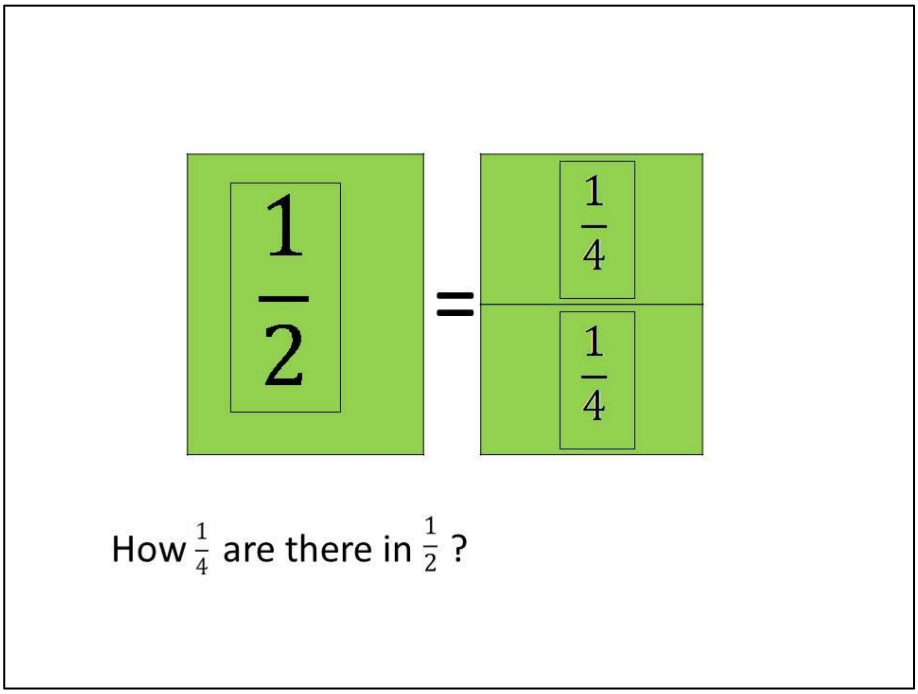
Slide 2



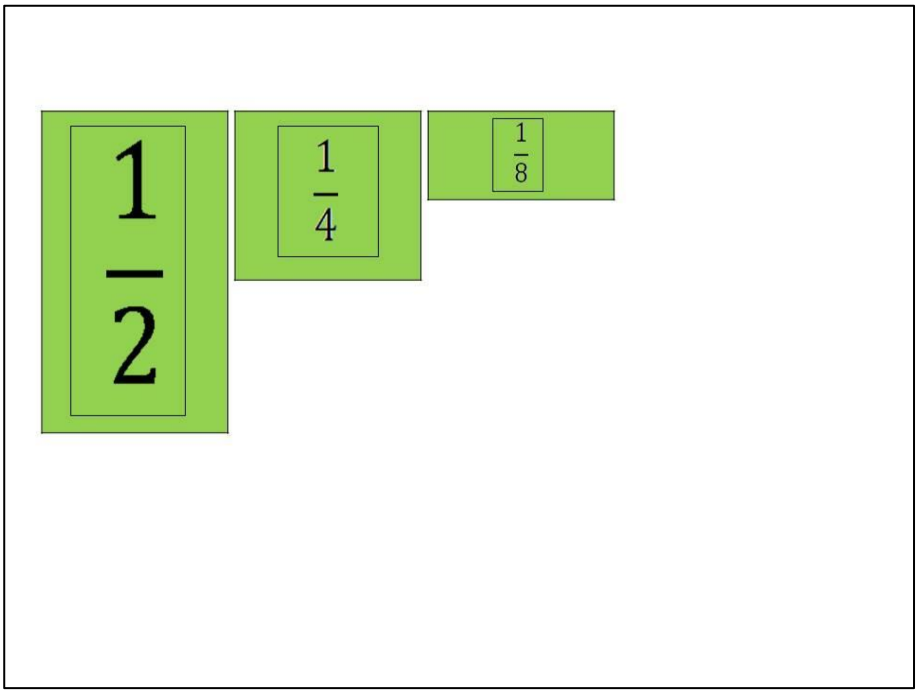
Slide3



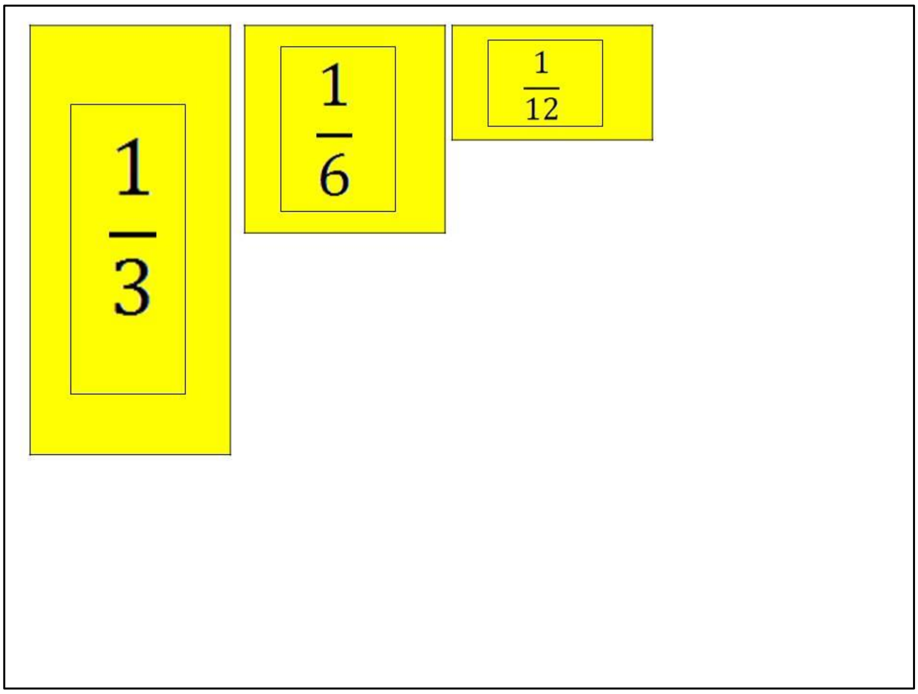
Slide 4



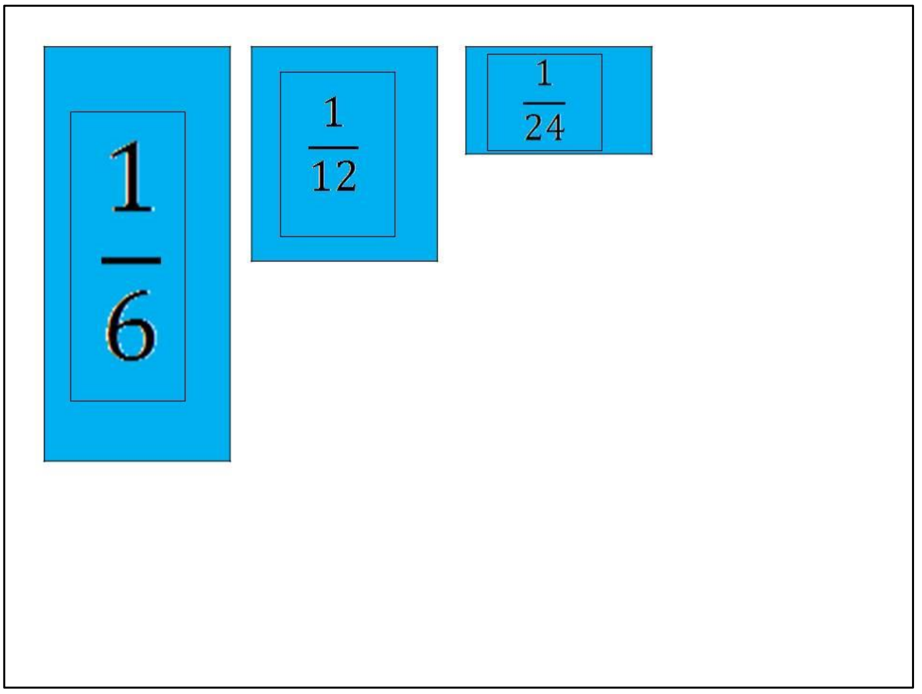
Slide 5



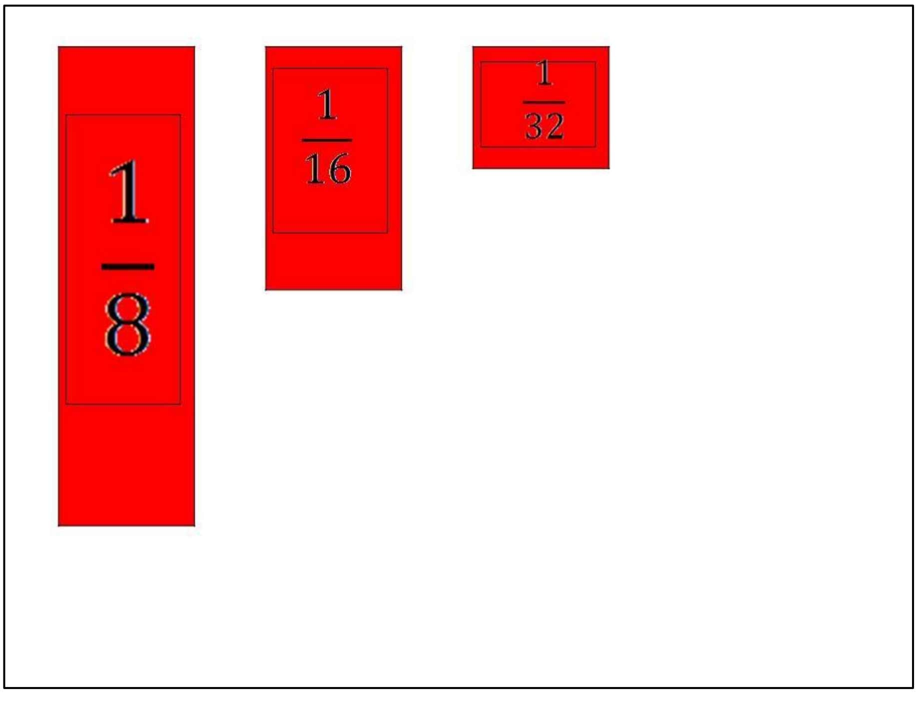
Slide 6



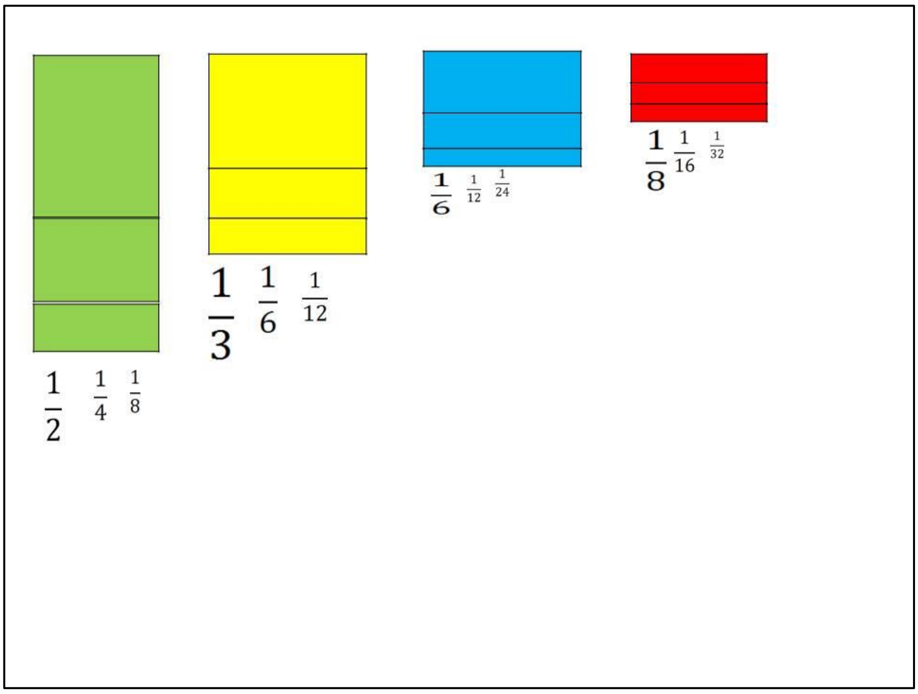
Slide 7



Slide 8



Slide 9



Slide 10

How Many

- How many $\frac{1}{4}$ are there in a $\frac{1}{2}$?
- How many $\frac{1}{6}$ are there in a $\frac{1}{3}$?

Worksheet 2 and Worksheet 2a

Slide 11

Writing mathematically

- How many $\frac{1}{4}$ are there in a $\frac{1}{2}$? The answer is 2

- If we were doing how many 5 are there in 15 the answer is 3. So we write it as $15 \div 5 = 3$

- So $\frac{1}{2} \div \frac{1}{4} = 2$

Worksheet 2b

Appendix 10 – Study lesson pupil survey questions

My help/views for the researcher Name ....

1. **When you worked with the fractions tiles how did they help you understand the fractions?**

2. **How easy did you find making up your own "What if Questions"?**

3. **How easy was the last task finding the answer to the concentrated quash problem?**

4. **Most of the lesson involved you working with another pupil. How did this help you understand fractions?**

5. **Which part of the lesson do you need to do more work on?**

Appendix 11 – Post lesson conversation – 1

With teacher C and after the filming of lesson with 7AC.

Me : Hi Teacher C – thanks for letting me use your class / classroom to do the work/ research.
Would you like to talk use through the lesson please?

Teacher C: Yes, It was a lesson on trying to see fractions within fractions; so dividing fractions.

Me : The class had obviously met fractions before.

Teacher C: Yes, Yes. It we are probably talking about maybe six weeks ago when we were adding and subtracting fractions. It was on their scheme of work to look at multiplying fractions. Whilst I was doing multiplying I talked about dividing fractions. We certainly talked about the trick; the trick – taking the reciprocal, doing the leave it, change it, flip it, sort of idea. But I also tried to get them to understand the reason why were flipping was it because we were doing a divide of a divide and it becomes a reverse operation.

Me : So you would normally go through that idea of the inverse operation where division would become a multiplication.

Teacher C : Yes, Yes certainly because when it comes to solving equations and you want to use the chain method [equations as function diagrams] you need to understand the reverse paths. I like to do it that way and take things logically step by step; and that can take a bit longer, but I think it is better to do the teaching for understanding rather than the trick. Ultimately you do end up showing them what the trick is, but you hope the understanding is there before you explain this is all that you are doing.

Me : What was interesting was they didn’t resort to the trick.

Teacher C: No they didn’t, but I don’t think they understood until about three quarters into the lesson that they were actually doing division of fractions. It just didn’t occur to them that they were dividing fractions; did it?

Me : No

Teacher C : As far as they were concerned they were seeing how many times does this fraction fit into another fraction.

Me: Yes, and it was probably only when the slide came up 15 divided 5 = 3; so what is an 1/8 into a ¼ that they made the connection.

Teacher C: Yes

Me: The lad that came up to the board (IWB) at the end. Talk

		me through what he said. [Talking about the final task]
39	Teacher C:	He broke it down didn’t he; so he has gone back to this logical idea. To be fair he started with a whole and eights and said how many eights in the whole; then this is ¾ how many eights is that and he explained that to me. Then he put them back together.
43	Me:	Do you there is a difference between the skills based approach of the ‘flip’ method and what I was trying to do with them today which was teaching for understanding?
46	Teacher C:	I think the understanding is going to take a lot longer, that’s the thing.
48	Me :	Do you?
49	Teacher C:	Yes; and I am not sure even though by the end of the lesson they could see it was a divide that any of them were actually thinking about dividing that fraction. I’m not sure they were actually dividing that fraction; they were trying to see how many times a fraction went into – I don’t think they made that link between dividing and what we were actually doing.
54	Me:	That’s probably OK because is probably the preface for a method later on. So this is probably the way into explaining division of fractions so as to be able to justify a method later on.
57	Teacher C:	Yes, I am just a bit worried .... you do need more lessons like this, but the problem is that they need to understand when they get a question on the examination paper that is saying ½ divided by 1/4 they need to understand how to do it and I not sure at the moment whether they would make that link.
62	Me:	It is interesting the two girls by the window had made up a question of their own (1/3 divided by 1/9) and they had got 3 as an answer, almost immediately. So I asked what is 1/3 divided by 2/9?
66	Teacher C	Right and what happened?
67	Me :	Immediately they said we have to the 1/3 into ninths.
68	Teacher C:	Right, so they are bringing it back to the adding and subtracting – to make equivalent fractions.
70	Me :	Yes; and then they said “so the question is now 2’s are there in 3”? It appears that they automatically said 9 divided by 9 is 1, so we need 3 divided by 2.
73	Me:	Interestingly they couldn’t do 3 divided by 2.
74	Teacher C :	No

75 Me : Until of the pair said – “well it is just 1 and a half” and it appears they got the solution on their own by using the equivalence method.

78 Teacher C : Yes

79 Me: The lesson was founded on skills, exercises, activities and tasks.
How did you think that worked?

81 Teacher C : Yes, I think the activity worked well because they were kept busy and they were working out problems for themselves.

83 Me : How do you see those 4 things linking together; or don’t they link together?

85 Teacher C : I don’t really know as they are really one and the same thing to me. There was definitely a skill at the beginning; I think it was a skill as they either knew it or they didn’t- it was a given.

88 Teacher C : It did flow as they used the skill for the exercises which they then used for the activity. All of these then linked to the final task.

90 Me : So they applied the learning to the task.

91 Teacher C : Yes, they did apply the learning to a task, but it comes back to whether they were dividing. Jason split up his one into 8 eights and his ¾ into 6 eights. So he has done as an addition of the number of eights

95 Me : Do you think there is a hierarchy in skills, exercises, activities and tasks? Or do think are all equal?
What is your view?

99 Teacher C : Well, my view in a general mathematical sense of teaching is that I would prefer we learn the skill and then make it much more functional and use the skill in a functional context. But, I m aware that the sticking point, and I keep coming back to it, is that to do that way takes much more time than we really have.

104 Me : Yes, I agree – I don’t think you can keep revisiting topics if you adopt this method.

106 Teacher C : Yes that is a problem; I think it comes down to how the individual how you learn. Some people are quicker and don’t really want to know a reason why, whereas other want to know why they do that. Why do we do this way rather than that way? Even if they do understand or they do the trick. For me I certainly want to know why that works.

112 Me : Yes, there are lots of children who want to know why that works and don’t always ask the question.

Is there anything you would like to add?

114 Teacher C : It was a good lesson and I will be using it in the future – however I do have to work out in my own mind where it fits into the scheme of work. It isn't a 50 minute lesson – it is certainly longer. I would also need to work out the next lesson in the sequence.

Appendix 12 – Post Lesson conversation 2.

With two teachers D and I after the filming of the lesson with 7AC

001	Me :	Which of the activities, teacher D, do you think the children engaged with most and why?
004	Teacher D:	I think the activities matching up the squares to the physical fractions, so finding the equivalent fractions half and quarter, I think ones that were struggling; it was accessible to all, they could take it as far as they wanted. It was a kinaesthetic activity.
011	Me :	What do you think Teacher I?
012	Teacher I:	I think they really enjoyed using the little fraction cards especially even in the later tasks, they were still using them to work out how many eights in a whole and how many eights in a quarter. They said they really enjoyed using them.
018	Me :	Did they?
019	Teacher I:	The girls, as you'll see on the video, said little cards as it made it easier for them to see.
022	Me :	Do you think they discovered the learning for themselves or do you they knew some of it already?
025	Teacher D:	I think they knew some of it already, but when it is presented in a different way. I think everyone did learn something – everyone did learn something about fractions.
030	Teacher I:	The girls here (pointing to a desk) were still asking how many eight in a whole, they were still putting them in, oh there are 8, it is 8 eights. So then they did twelfths, so in one whole there are 12 twelfths. So we haven't got enough twelfths for 2 wholes.
036	Teacher I:	So if this one has 12 twelfths how many do this whole have? Oh, it is going to have 12 as well. It took a while for that to sink in? I thought was quite a strange thing.
041	Teacher D:	At the beginning they automatically knew 0.5

		is 50% is a half that was embedded. But actually seeing and doing it is very different.
044	Me :	The lesson was about
		(pause)
046	Me:	What?
		(pause)
048	Me:	I didn't give them lesson objectives, but could you pick up what the lesson objectives were?
050	Teacher D:	To understand equivalent fractions.
051	Teacher I:	They were dividing fractions
052	Me:	So the lesson was about using equivalent fractions to do the division of fractions.
054	Me :	I didn't tell them the trick of turning the fractions upside and multiplying. They came out of the lesson with a different approach. So, do you think that was a better approach, in terms of the way you would normally teach the topic, or the way you were taught.(teacher I)
060	Teacher I:	I normally the method of flipping the second fraction and multiplying. Last year I had a top set and they asked why? I did show them why with some difficult numbers. I think for KS 3 this method is a brilliant way and I will use it, and it is a very different way of doing it. I am not sure it would work for KS 4 they might think it was not too 'babyish' way but the want of a better phrase when messing with little cards.
069	Teacher D:	I think it explains to them what you are asking
070	Teacher I:	Agrees
071	Teacher D:	Because when you give them a sum it is just a sum to them, but when you do it that way (indicating the way it was taught in this lesson) they are actually understanding what the sum is and they are looking at how many eights go into one whole and three quarters

(talking about the last activity), whilst doing the division

078 Me : The actual lesson included some skills, exercises, activities and a task.

Do you think there is a difference between skills, exercises, activities and tasks?

082 Teacher D: I think they are all interlinked with each other?

083 Me : Do you?

084 Teacher D: Yes

085 Teacher I: I agree with (teacher D).

086 Me : Do you think that if I had started with the tasks straight after the tiles then the lesson would have still worked?

(pause)

The very first thing on the IWB was two halves make a whole. If I had then presented the task this is the problem we are trying to solve (indicating worksheet 3). Do you think the class would have been able to have solve the problem?

095 Teacher D: Some of them.
(pause)

097 Teacher I: Agrees

098 Teacher D: The structure of the various phases of the lesson (skills, exercises, activities) allowed them to access the task. It is difficult to know if any of them would have been able to make the link from the skill straight to solving the task.

103 Me : It was interesting the comment you made – they were all interlinked. Interlinked to me means joined. Is that what you are trying to imply?

106 Teacher D: Yes

107 Teacher I: They were in the ‘right’ sequence – they all followed on. They skills, exercises, activities and task seem to be joined rather than interlinked.

110	Me :	My notion of what I was trying to investigate was that they are not interlinked but embedded within each other. So a skill is the lowest level, then the activity – writing a fraction mathematically, then the task is a much harder problem. So it is almost like an inverted pyramid of difficulty.
117	Teacher D:	It is similar to scaffolding
1118	Me :	That is a good analogy
119	Teacher I:	It is like a progress ladder
120	Me :	Yes, it is like both of these – where skills are a subset of exercises, exercises are a subset of activities and activities are a subset of tasks.
123	Me :	Going back to what you were saying Steph about the 'flip' method. Where does this method sit in this system?
126	Teacher I	It is just a skill, the 'flip' method is just a trick
127	Me :	Do you think the lesson built understanding of division of fractions? If so why?
129	Teacher D:	Yes. Definitely. Instead of just learning the 'flip' method they learnt why they were doing the division of fractions. They were learning why and how they were doing the division. They were looking at how many eights go into one and three quarters.
135	Me :	The wording is quite deliberate, because division in this context is 'how many'. There was a pedagogical reason for the wording. Do you realise this?
138	Teacher D:	No, not a first – it was much later on in the lesson.
139	Teacher I:	Some of the group did because they knew how many eights went into one and they knew it was eight. They knew it was divide. They were almost filling up the whole with eights, but not 'adding it'. They were doing it in their head.
144	Me :	There were two groups – the girls over there that actually got the end of the task and started to write their own questions. Were you surprised by that?

147 Teacher I: Yes I was- I haven’t seen this group before but when I was looking at their work of the boys and girls here (a different group) both were getting it. But when the girls over there (the group in question) were writing their own question I hadn’t been over to them – I was surprised as that was really very quick. I thought they did it very quick in comparison to some of the others in the group and when it has previously been taught.

156 Teacher D: Yes, I agree

157 Me: Do you think the social constructivist learning approach, which was at the heart of the lesson, where the pupils were constructing the learning; was the right method to teach this topic?

161 Teacher D: Yes, I think the discussion helped them build on the knowledge. So, they were certainly helping each other.

164 Teacher I: They were definitely sharing their ideas with each other, explaining to each other and prompting each other.
So I was able to ask:-
How did you get that answer?
What did you do?
And they gave me full on explanations a couple of groups.
They were saying with did this and this and we got the answer this, and I asked why.

174 Me : Were you surprised how they just worked together on the problem as you know the children in the school much better than I do?

177 Teacher D: I think it is just common practice.

178 Teacher I: Yes it is common practice, but they were sat by a friend.

180 Me : Do you think is important?

181 Teacher I: I think it was, if you had put them ‘boy-girl’ seating that might not have worked so well. I don’t think they were sat in their normal seating plan .. simply because we moved some of them to make pupil pairs. I think if you had put them in a ‘boy-girl’ pair they wouldn’t have done as much discussing. There was one girl who wanted to work alone.

188 Me : Do you think the lesson worked for her?

189	Teacher I:	Yes, the activities could access in as pairs or as an individual.
191	Me:	What comments would you make about the learning and the teaching?
193	Teacher I:	I think there was a lot of pupil led learning. You drew a lot of the learning out of them rather than you saying this is how you do division of fractions. You got them to say this how you this. When one lad said the and answer is a 1/3 when actually it was 1/12, you got the others to explain why it was a twelfth and what is happening.
199	Me :	Would you do that lesson?
200	Teacher D:	Yes – definitively
201	Teacher I:	Yes – definitely
202	Me :	That way?
203	Teacher D:	I would spend much longer ‘fiddling’ with the bits (indicating the manipulatives) the kinaesthetic part.
205	Teacher I:	Perhaps do a lesson on those.
206	Teacher D:	Yes – a whole lesson on doing that and then go on to the task in another lesson.
208	Teacher I:	Yea, depending on the group. If it was set 1 I would do the whole lesson in a lesson.
210	Me :	The lesson had 4 distinct parts activity, skill, exercise and a task. Is this a better structure than the 3-part lesson?
213	Teacher I:	Yes. However, to get all four elements in one lesson for every topic is probably not possible. Additionally the lessons then just move from 3 parts to 4. This is not really giving variety for the pupils.
217	Teacher D:	I wouldn’t want to do 4 parts either because some aspects of mathematics are really difficult to find real-life tasks. Some are studied just for the beauty of the topic.
221	Me :	Was this set 2?

222	Teacher I:	Yes, with set 2 I would definitely go over the 2 lessons. Yes some of the other coloured cards (There were other fraction cards which were not used in this lesson). Knowing your own set also you would probably have to give some pairs slightly different questions. Certainly some groups might have needed the other fraction cards.
229	Me :	One last question. The pupils have a notion in their brains of how they divide. They normal have a scheme of how they do division or addition for example. The 'flip' method doesn't use that scheme.
233	Teacher I:	It doesn't use divide
234	Me :	No it doesn't divide at all.
235		Do you think this of building from something they know about equivalent fractions would allow us to use that division map they already have in their minds?
239	Teacher D:	At this level, I think, well; it is a bit longer.
240	Teacher I:	Yes I suppose
241	Teacher D:	A lot of the children were understanding the questions.
243	Teacher I:	A lot of children with divide will think 27 divide 3 will do 3, 6, 9 they will almost do the times tables. So I suppose if they knew that one and half was 12 eights. They could go that I have 4 eights oh that's a half and I have 4 eights that's a half so perhaps they might be able to do it the same way.
249	Teacher D	Yes
250	Me :	Certainly equivalence is the basis of this work and a number of children were using multiplication as a basis for the work. It is very easy to slip into multiplication (this is not surprising as the operations are inverses of each other).
255	Me :	Criticise the teaching now.
		(pause)
		Did the teaching prompt the learning?
258	Teacher D:	I think it did because you used leading questions

and you asked a variety of pupils and you were asking for explanations rather than just the answer.

261 Me : The lesson didn’t follow the normal pattern of a lesson.

263 Teacher D: No

264 Me : No,

265 Teacher D: It didn’t have a starter or plenary.

266 Teacher I: It didn’t have any objectives

267 Me : Was that a problem?

268 Teacher D: No it didn’t negatively impact on their learning – maybe just more difficult to measure their progress. So it might be not clear which pupils fully understood – but we will get from the video footage.

272 Me : And the last activity as well.

273 Teacher D: Yes

274 Me : If you were to do that lesson – how would you alter it?

276 Teacher I: There was really a starter – “What do you know about fractions?” I would have given this back at the end of the lesson – so what else do you know about fractions now (in a different colour).

This would have allowed you to do the 3-part lesson as you are supposed to do.

282 Me : Teacher I – when you completed your GCSE as part of the assessment you had to write up a number of pieces coursework. How do you think the parts of this lesson relate to your school of mathematics?

286 Teacher I: I remember my mathematics really well. We had to practice lots of skills from textbooks before we could do the coursework. I remember one coursework involved having to expand brackets – in the days before we were given the coursework we probably did two hundred or more (or it seemed like it at the time) expansions. I thought this was a really good idea. I have now come to

the belief that large exercises of skills type questions are rarely help pupils and it is not a method of teaching that I use frequently. To some extent if I had to teach a skill I would prefer the approach as in this lesson..

298 Me : Teacher D – when you completed your GCSE all coursework had gone from the assessment. How do you think the parts of this lesson relate to your school of mathematics?

302 Teacher D: Well I remember my mathematics lessons really well. It has only been 7 years seen I sat GCSE's and we didn't have any course or extended tasks. Our mathematics curriculum was much more problem based, we didn't have many lessons where we practised loads of questions. This lesson sort of replicates the style of my school mathematics lessons. We were often given a single problem and in pairs we would have to find a solution – sometimes as a starter we would be taught a skill which was needed for the problem but not always. I don't think skills or exercises should be the most important things we do in mathematics lessons.

315 Teachers D and I: Can we have the tiles and the worksheets please!

316 Me : The mathematics of dividing two fractions didn't use the normal accepted approach. What are your thoughts?

318 Teacher D: Well, I have never seen this method before and I wasn't sure it would work mathematically or if the pupils would understand what they were being asked to do. I guess I was relying on your knowledge gained over the years and that you would have tried this method before. My knowledge of mathematics, because I did the subject knowledge enhancement course (SKE), is developing all the time. I am expecting it to improve over time as I gain experience and teach across the ability range.

327 Teacher I: I have seen this method of dividing fraction before, my mentor in my first practice at XX school used it. I didn't understand it and I didn't ask him for an explanation, mainly because I was afraid he would mark me down on the subject knowledge teaching standard. He had a mathematics degree and I assumed he would

know best. I’m like XXX (teacher D) I also did the SKE course and this approach was not taught on that course. My mathematical knowledge is improving but the experience of teaching a top set year 10 this year has made me research topics in detail. Experience, as XX (teacher D) says is the key.

The next part (100+ lines) of the interview have been removed for brevity.

451 Me : Today’s lesson differed from the standard 3 parts that the school uses. What are your thoughts about this?

453 Teacher D: Yes, you had 4 parts which you call learning episodes. We have to start by telling the children what the objective of the lesson is and make his plain for pupils working at different levels which we label gold, silver and bronze. You just said “today we are learning about fractions”. I found this strange.

458 Teacher I: It made me think. I wasn’t sure I liked this approach, I wondered if the pupils would learn anything.

I like the 4 parts. I think this gives me a freedom that the 3 part lesson doesn’t and on the basis of what I have seen today I think the pupils did, mind this might be because you are different teacher and they were using the tiles.

464 Teacher D: I’m not saying I didn’t like the structure of the lesson just the opening of the lesson was very strange.

466 Teacher I: While the lesson was going on – I thought this is just a 3 part lesson with an extra part. It was only towards the end when I realised that all the parts were interconnected and building up to being able to solve the task problem. This is very different to our lessons which have three parts and the parts are very often not really well connected. Especially the first part which tends to be just some arithmetic to get the pupils ready for the main part of the lesson.

474 Teacher D: Yes I think you are right (Teacher I), I liked the way the activity created the key words and

finding out what the group knew about fractions which then moved straight into the exercise and skills part of the lesson, and these were then used to solve the task. I could see this approach working for lots of topics but not all.

480 Teacher I: I also think the sequence of activity, exercise, skill and task is clever and might be the reason why this low ability group were able to do so much of the work.

482 Me : That’s an interesting comment. What do you mean?

483 Teacher I: Well the four parts of the lesson seemed to link to each other and they weren’t disjointed like some of the lessons we do. The activity at the beginning with pupils working together recalling what they know is something I would do but at the end of the lesson to check learning.

488 Teacher D: And me … but it is more logical to have it at the start of the lesson and the pressure of sitting in quiet do some arithmetic at the start of the lesson is no longer there.

What I thought was more interesting was the exercise and skills where they had to make up their own questions. I would never do this. I would worry they would make up questions that were silly or ones they couldn’t answer.

495 Teacher I: I wouldn’t do this – but it worked and the group really took control of what they wanted to learn. This was really interesting and something I might try next lesson.

498 Me : The task at the end of the lesson. What are your thoughts about how this worked?

500 Teacher D: Well it certainly worked they all produced a correct answer in a relative small amount of time. The tiles helped, but having the freedom to create their own problems in the exercises and skills gave them the confidence to solve the task. What was even more interesting was they were able to make up their own task problems which were wordy problems, and then solve them. I would never have guess they could have done this.

508 Teacher I: I don’t think the tiles were they important in

solving the task, their confidence and self-belief that could do the questions was probably more important.

The next part (200+ lines) of the interview has been removed for brevity.

712 Me: Returning to the lesson structure. In the first part of the research lesson the pupils were doing an activity where they were recalling information about fractions. How do think this worked?

715 Teacher I: It was clear that the activity engaged the pupil pairs and for me your definition of an activity as recall of knowledge with no new learning was absolutely plain. It is an approach I will use. It is a break from what we normally do, but the engagement levels were good and the amount of facts the group were able to recall was more than good. I think this is a good way to teach maths but the school restricts us so much in the way lessons should look it makes this type of opening activity very difficult.

723 Me: In the second and third episodes of the lesson pupils were devising and answering their own questions. Do think this is a good approach?

726 Teacher D: I thought the pupils would be disengaged when you did this. In actual fact they were more engaged. The quantity and quality of the questions and discussions was really interesting. I was amazed at the amount of work certain pupils did, it was completely different to what they normally do in class. I expected them to write some really weird questions that could answer – but this didn't happen. I don't think approach would work with every topic and maybe not every class.

734 Teacher I: I liked the approach but I don't think I would have the confidence to do this. I certainly find it difficult to move away from the school approach.

There were 78 more lines of the interview these have been removed for brevity.

Appendix 13 – Post Lesson conversation 3 –
With three teachers D, H and I after the filming of the lesson with 7NR

001	Me :	Would you like to say what you thought the differences were between the two sessions, Teacher D.
004	Teacher D:	I thought the difference with today's was that the group were doing the work more numerically – the times and dividing rather than actually using the fraction cards to work it out.
008	Me :	So you think they did need the fraction cards?
009	Teacher D:	No they didn't, probably because they were set 1 today as compared with set 2 from the previous lesson.
012	Me :	What did you think the difference was Teacher I?
013	Teacher I:	Yes, they did seem to use the fraction cards a lot less than the other group. Straight away they were saying 8 eights is a whole one and if I times by 2 ¾ becomes 6/8. They were getting it very much more quicker than the set 2's
018	Me :	Teacher H, you did see the previous lesson. What did you think of the lesson?
020	Teacher H:	There seemed to be a lot of deep thinking going on today.
022	Me	Do you think the tiles were useful?
023	Teacher D / Teacher H:	Yes (together). When they weren't using the tiles they seemed to get things the wrong way round; to get from a 1/3 to a 1/6 you times by 2.
024	Me:	What did you think about the skill, exercise, activity and task? They were the same ones as the previous lesson.
027	Teacher D:	They seemed to go quicker. They got the concept (equivalent fractions) from the skill. They got the understanding of the division of fractions much quicker; some of the pupils in the first lesson didn't realise they were dividing fractions, but they all realised it in this class by the end of the skill and the start of the worksheet exercise. They were very quickly writing out and making their own questions.
035	Me:	Remember the first sheet (the A3 sheet) where they

were writing what they thought fractions are; what are your thoughts about that sheet?

038 Teacher I: There was an awful lot more on them from the pupils this time. They knew what numerator and denominator were as well as knowing what a top-heavy fraction and improper fractions were.

042 Me: They all seemed to know the mathematical language, but few of the group came up with some of things like the other group such 50% = ½

045 Teacher I: A few were saying these sorts of things but weren’t writing them down. It was almost like something that you didn’t need to write down. It was strange – one said ½ Oh that is the same as2/4 and stuff like that – but only one wrote it down in the end.

They almost bypassed the ½ = 50% because .....

052 Me: Yes, I think you are probably right. The exercise, fraction pairs, where ½ = ¼ + ¼. How do think this went?

055 Teacher I: They were very quickly into making up their own .. like 1/96= 1/192+1/192. They didn’t do that all last week.

057 Teacher D The group I was look at last week struggled a lot more at that point in the lesson than the group I was with last week.

060 Teacher I: Once they saw it today they didn’t need to use the fraction cards.

062 Me : What was fascinating today was that a lot of the group went back to the cards to do the very last task – they put own on the table 1 and ¾ and started to look at the number of eights in the 1 and ¾. They didn’t do that in the last lesson.

067 Teacher D: They needed to be prompted in the last lesson

068 Teacher I: We definitely had to prompt set 2 to do that and we didn’t with set one. The table I was working/ filming immediately said there will be 8 eights in this one.

071 Me: What about the boy who stood at the board to explain 1 and ¾ divided by 1/8

073	Teacher D:	That was a very good explanation and it was exactly how he was explaining it to his partner. He did reword it and he could explain how he did it in several different ways. The way he split the whole and made the ¾ into 6 eights. He actually made the denominators the same.
079	Me:	That could be a result of the equivalent fractions or perhaps what they had already done with adding and subtracting fractions and making the denominators the same What about the girl who gave the explanation?
084	Teacher D:	That was brilliant – very mathematical literate. Apparently she hasn't been very forthcoming in previous lessons. She hasn't been very engaged.
087	Teacher I:	I think this is because a few lessons ago we were talking about pie charts and I think what she was going to say was right but then she stopped herself from saying it – the mathematics would have worked – she almost thinks a bit too mathsy in somethings because she was making it really hard for herself but it would have worked. I think that was interesting today the explanation she gave was probably one of the best I have heard recently.
097	Teacher D:	It was better than I could have explained it, and sometimes we need to listen to the children's explanations. It showed deep understanding didn't it?
100	Me:	Do you think the materials now that you have seen them twice helps to develop that deep understanding?
103	Teacher I:	Yes
104	Teacher D:	Yes
105	Me:	Would you use them to teach division of fractions
106	Teacher I and D :	Definitely – can we have copies please?
107	Me:	Not a problem – Teacher D already has the electronic files.
109	Me:	Are there any comments make about anything that I haven't covered and that you think are important.

111 Teacher H: I think the boy that came to the front to do his explanation; it shows that it is good to encourage that there are different ways to get to the same answer. It was good to make sure that even though everyone was going down the one route of using the tiles to get to the answer to get how many eights there are in a whole. He had a completely different spin on it and it was still correct.

So it is nice to show the others that there is always more than one way of doing something.

You also have to be quite happy to let pupils come up and do that even though that wasn't quite what the lesson was about. Whether he did or didn't use the tiles to get to his solution he had an alternative spin on the question and what he said also works all the time, but I don't know where he got his method from.

127 Me: Do you think the materials and the way the lesson was done would develop division of fractions as a better way than we would normally teach this topic?

130 Teacher D: Better, because it develops the understanding rather than just a process that you have to do. It is obviously more memorable if they know and understand what they are doing.

134 Teacher I: I think it would – but I might go back to the flip method; probably; it depends; I don't know.

136 Teacher D: I don't think I will, no, because they always forget to flip and times so I think I will use this method.

138 Teacher I: I think it sometimes depends on the question. With a worded question like that I think I probably would probably do it this new way. If was written as a normal divide then I would use the flip method. The ones that are used in exams always work out to be nice answers, so why should we make things difficult when we can find a way that works.

145 Teacher D: What was interesting was the two girls that wrote down 1/3 divided by 2/9 and she changed that all into ninths and ended up with 3/9 divided by 2/9 and immediately said the denominator is 1, but they couldn't work out what 3 divided by 2 was. So they couldn't write down that the answer was one and half. It took a little while to get the one and a half.

Even awkward questions they were able to see that they could be done using that method.

154 Me: I think that would probably be the next lesson?

Any final comments.

156 Teacher H: I think they would have been happy to stop another 50 minutes. I also think they really engaged and I had some comments at the end of the lesson saying that they really enjoyed the lesson.

160 Teacher D: One lad would have like some more challenging questions.

161 Me: The lesson was in 4 parts. This is different to 3 parts we talked about during your training. What are your thoughts on moving to a 4, or more parts lesson?

163 Teacher H: I liked the idea of a 3 part lesson – it seems very tight if a little prescriptive. We have had lots of conversations with (subject leader 1) as she is a real advocate of this structure. I had started to think we should move away from it and more towards the structure of the lesson we have just seen. It would then allow me the freedoms to think about activities, skills, exercises and tasks. However at the moment I feel I have to follow, and move more towards what the subject leader is looking for, especially as she will be doing my performance management.

172 Teacher D: I like the 4 parts in this lesson it gives me the freedom to think about lessons in a very different way. I have a different subject leader who is equally convinced that 3 parts (starter, main and plenary) is the best and possibly the only way to structure a lesson.

175 Me: So after this lesson what are you views about how a lesson might be structured?

177 Teacher H: I think changing my view will be extremely difficult not because of some deep philosophical or ideological view it is more about keeping the ‘party – line’ and being a team player. The more time that passes and the longer you become a member of the team the greater the difficulty I think it will be to change. As I said I like this lesson and the structure and the freedoms it allows but I probably will stay with the 3 part lesson so as to be seen as a team player.

185 Me: Was it evident to you what the differences were between the 4 learning episodes?

187 Teacher D: Absolutely, I have a very different view of what an activity is now. Your version of a tasks is not really very different to what I envisaged and it is shat you explained to us when were training. Your definition of an activity is clear and precise and I think this is closer to the real meaning than what I normally write in my planning.

193 Teacher H: I agree, but the skill and exercise episodes are very different to what I would have said. I think the idea of letting pupils make up questions is scary – but I can see the advantages and it is definitely something I will try in the future. I agree also that your definition of a skill and an exercise is very different to the way I use the terms.

There were 151 more lines of the interview these have been removed for brevity.

Appendix 14 – Transcript of the final part of an interview.

With a national figure in the key stage 3 mathematics strategy.

A definition of skills and tasks was shared before the interview (these were those used on the questionnaire).

Me :	What types of tasks could support better pupil understanding of mathematics?
Interviewee:	First of all in mathematics I think a task has to be hook in to it and useful and a purpose behind it. There are two types of task to me There is a conclusion so that you make a connection back – something with and end product and that is tangible. The other one is the more old fashioned one of simply "Awe and Wonder", which is something that triggers them in to "What If" they choose a more dynamic way of looking at mathematics. "Wow" so what if I explore this in a different way or change the parameters what would happen next. So it is making them develop their thinking skills develop.
Me:	So are you think of functionality and Pure mathematics
Interviewee:	Yes that's right I am and the balance between the two. If they get too much of one then they lose the finesse of maths and if they get too much of the other there is nothing tangible to talk about maths.
Me:	So do think that applies to all children or just some?
Interviewee:	I think you have to relate it back to thinking skills. Again, from my world; what I would say is that once a child has moved through what I would call a confident level 5 student; they have high level thinking skills; and that is when you can accelerate their learning in terms of investigating and exploring and enticing them into more and more. But before that it is much more about enjoy but more concrete, but then again that is just my perception.

Me:	Do you think there is a place for skills as well, using this definition of skills
Interviewee:	There is a need for skill because they are more confidence and foundation because some things have to be learnt, they have got to have understanding in my book to be able to challenge themselves and justify why – because that has got to be skills based, and if have a whole load of skills then can use those skills to solve problems / tasks. Skills and tasks is a bit like chicken and egg with tasks and skills.
Me:	Do you think there is more of one than the other?
Interviewee:	What is happening because of the pressure of league tables and settlers; what happens is that people are cutting things short and I would say what we are seeing is tricks. Which is really worrying but do have a heavy weighting on skills and this is what I am really glad because actually get back to children thinking.
Me :	My second question is. What apparent mathematical misconceptions and barriers prevent pupils learning?
Interviewee:	(Long pause for thought) Misconceptions I would say come from poor teaching or come from just passed down from families. What happens next is that we present a question that then does not work. A misconceptions like the other day we were talking about a function and we had to go back to logical thought work and go back to a proof and let logical thinking to work to iron out something that they had assumed they no one had taught them. So that they could draw a function.. If they have gaps in their knowledge we go back fill in gaps with something that makes sense in there world.
Me:	Thanks very much.

Appendix 15 – Class Profile of the study classes

	7NR			7AC	
	Raw Numbers	Percentage		Raw Numbers	Percentage
Class Size	21			31	
Females	11	52.3%		14	44%
Males	10	47.6%		17	55%
Mathematics Profiles					
On entry below national average	19	90.5%		0	0%
On entry at national average	2	9.5%		25	80.6%
On entry above national average	0	0%		6	19.4%
At study time below national average	15	71.5%		0	0%
At study time at national average	6	28.6%		18	58.0%
At study time above national average	0	0%		13	42.0%
Reading Profiles					
At study time below National average	21	100%		20	64.5%
At study time at national average	0	0%		8	25.8%
At study time above national average	0	0%		3	9.7%
Pupils with special needs	5	23.8%		2	6.5%

Appendix 16 – Study School Lesson Plan Proforma

Date:	**Subject:**	**Period:**
Year Group:	**Setting:**	

What is the teacher doing to engage students in their learning?	Timings	What are the students doing? How is the lesson differentiated?
1. Prepare for Learning (settling and recapping)		
2.1 Agree Learning Outcomes (use the learning objectives slide and make reference to Bloom's taxonomy to ensure appropriate levels of challenge) **BRONZE:** **SILVER:** **GOLD:**		
2.2 Literacy Focus (use the half termly literacy focus or information from marking and assessment to inform the lessons specific literacy focus)		
3. Presenting New Information to students		
4. Active Learning (students to process info and make sense of it. What are they doing with the new information to develop their knowledge and understanding?)		
5. Demonstration of New Understanding		
6. Review (have learning outcomes been achieved? If so, how? If not, why not? Develop student ability to reflect upon their learning and how they can improve)		

Appendix 17 - University lesson plan proforma

Institute of Education: Secondary Mathematics - Lesson Plan Overview

Trainee:	Teacher in charge:	School attachment: please circle 1 2	School:

Class	Date	Number of pupils	Time/From -To	Room	NC / GCSE Target grade

Topic (inc. New National Curriculum ref).		Learning Objective(s)	

Learning Outcomes – what the pupils should know, understand and be able to do by the end of the lesson	
All	
Most	
Some	

Pre-requisite knowledge		Use of ICT support?	

List main teaching/ learning strategies		Literacy and/or Numeracy issues	
List main questions			
Sources of planned assessment evidence Ways of obtaining feedback on achievement of learning outcomes during and after lesson including homework	*Q and A* Classroom monitoring Routine observations Marking books Direct intervention List others:	Homework: Date due in Additional (i.e. contribution to personal, moral, social, and cultural development) :	
Resources required			
Safety issues Risk assessment made?			
Your personal targets Please refer to the DfE 2012 Teachers' standards as appropriate			

Lesson Plan: Timing of activities - This should make clear the phases of the lesson, including KEY QUESTIONS at relevant points. The layout of of the plan should make it easy during the lesson itself.

Class:		Date:
Time	Pupils' activities	Teacher's inputs and activities
0 to 5 mins		
5 to 15 mins		
15 to 30 mins		
30 to 45 mins		
45 to end of lesson mins		

Appendix 18 – Research Study Lesson Proforma

<table>
<tr><td>

Title:

Lesson Objective

</td><td>

Thinking strands

Curriculum Links

Resources

</td></tr>
</table>

Lesson Summary

1. **Learning Episode 1 (Timing)**

2. **Learning Episode 2 (Timing)**

3. **Learning Episode 3 (Timing)**

4. **Learning Episode 4 (Timing)**

Mathematical Content

Pupils' thinking

Appendix 19 – An example lesson plan

From teacher C prior to the research.

Lesson Plan: Timing of activities (This should make clear the phases of the lesson, including KEY QUESTIONS at relevant points. The layout of the plan should make it easy to use during the lesson itself)

<table>
<tr><td colspan="2">Class:</td><td>Date:</td></tr>
<tr><td>Time</td><td>Pupils' activities</td><td>Teacher's inputs and activities</td></tr>
<tr><td>0 to 5 mins</td><td>Multiplication grids – write up WALT and WILF and date</td><td>Complete the exercise on the worksheet.
WALT and WILF in books - make sure pupil A and Pupil B do this.
Let them complete the exercise and pupil copy up the first 2 on SMART - self assess.</td></tr>
<tr><td>5 to 55 mins</td><td>Explanation on ratio – finding and simplifying ratios. powerpoint presentation.

Activity 1:
Worksheet exercise on ratio to peer assess on powerpoint.

Extension Task: on back of sheet, to self assess (pupils) on SMART. Demonstration of neck "bling" activity – teacher demonstrates.

Students given an envelope with a ratio, string and cubes: e.g. Make a green and red necklace in the ratio 2:3 They do not tell anyone what their ratio is and after the 5 mins where they have to make their necklace, they come to the front and the others have to write down what ratio their necklace is in. (to assess)

Hint to them as they 'model' them i.e. note down the number of blue and the number of green. See if you can make a ratio from that.

After every student has made necklace, give out assessment sheet and explain how to use it - students present their "bling".</td><td>Complete on sheet peer assess task (ppt) - comment
Extension Task : on back. Self assess (SMART)
Student makes necklace with their differentiated ratio.

Students demonstrate their necklace in turn. Assessment multiple choice exercise on worksheet

Peer assess.

Match up activity and stick in books.
Complete on sheet peer assess (ppt) – comment

Extension Task : on back. Self assess (SMART)
Student makes necklace with their differentiated ratio.
Students demonstrate their necklace in turn.

Assessment exercise multiple choice on worksheet

Peer assess.</td></tr>
<tr><td>Plenary 5mins</td><td>What did we learn?

FUSE AFL.

Extension activity – simplifying ratio bingo on PowerPoint if plenty time.</td><td>Write in books.</td></tr>
</table>

Appendix 20 – An example lesson plan

From the study school produce by a female teacher. The bold is my emphasise to show the use of pedagogical terminology in the lesson planning.

Lesson Plan: Timing of activities (This should make clear the phases of the lesson, including KEY QUESTIONS at relevant points. The layout of the plan should make it easy to use during the lesson itself)

Class:		Date:
Time	Pupils' activities	Teacher's inputs and activities
0 to 5 mins	Pupils enter room and settle	Ensure pupils enter room in a suitable manner with appropriate uniform. Hand books out, and make sure pupils have all materials they need for the lesson.
5 to 15 mins	Write date, title and learning objective in book. Starter How old? Using the clues on the board pupils can work out how old the man is. Can use quiet partner voices to discuss problem.	Share learning objectives and outcomes, ask pupils to read each learning objective aloud. Communicate what is expected by the pupils by the end of the lesson. Ask pupils to work out how old the man is. Walk around class and give prompts to pupils who may be struggling. **Give five minute time limit to task.**
15 to 25 mins	**<u>Main Activity</u>** Theoretical Probability; Q. What does theoretical/ in theory mean? Q. How do we calculate theoretical probability? Experimental Probability; Q. What does experimental mean? Q. What comes out of doing an experiment? Q. How can you record results? Write how we calculate experimental probability in book. Example; What is the important information that Simon has gave us? Efficient way to record data? What does the experimental and theoretical probabilities tell us?	Discuss the meaning of theoretical probability- give examples from the work were doing with coins and cards last week. Explain that's it's based on what we think should happen. Recap how to calculate probability in fraction form. Explore and discuss the meaning of experiment and what this might mean for calculating probability. Explain that; We use experimental probability to estimate probability and make predictions. Calculated after an experiment has been completed. Tell pupils how we calculate experimental probability. Ensure they understand the difference between experimental probability and theoretical by thumbs up/down. Run through example with pupils, ask them to do the **exercise** calculate both probability's using the outcomes and data on the board. Ask pupils to use keywords to explain and draw conclusion from data.
25 to 40 mins	**Activity 1**- DICE; In pairs or threes use worksheet to conduct own experiment using dice. **Activity 2**- COIN/COUNTER In pairs or threes use worksheet to conduct own experiment using dice. **Extension Task-** experiment with counters or coins (whichever one they haven't done)	Explain that they are to calculate the theoretical probabilities first, then hypothesis what will happen. Then do the **1st Exercise** and calculate the experimental probabilities. The must conclude using keywords. **2nd Exercise**- differentiate between pupils who are confident can do counter experiment (more outcomes). Less confident pupils can do coin experiment.
40 to end of lesson mins	Plenary Pupils to answer multiple choice questions from the **exercise** on the board on their whiteboards. RECAP; Thumbs up and thumbs down. How did I do today?	**Exercise Recap -** Read out questions on the board as they appear, check all answers on whiteboards to ensure pupils are getting the right questions. Remind class of learning objectives Gauge how pupils have understood today's work, thumbs up/down. HOMEWORK; hand out and explain homework. Due Thursday.

Appendix 21 – Pre Study Lesson Plan from Teacher A

Lesson Plan: Bold is my emphasis – showing an inconsistent use of terminology prior to the research lesson.

Class : Year 8	Date : March 10th
Lesson Aims	Sharing in a given ratio
Desired Learning Outcomes - what the pupils should know , understand and be able to do by the end of the lesson	
All	Understand how to distribute an amount correctly, using a given ' whole ' and a ratio (a:b)
Some	Understand how to distribute an amount correctly, using a given 'whole' and a ratio (a:b:c)
A few	Calculate the size of remaining parts, given the ratio of parts, and the size of one part. (inverse calculation)

List main questions	How many total parts are there?

Time	Pupils' activities	Teacher's inputs and activities
0-10mins	Pupils enter room 5 - 10min on starter – **exercise** of 10 multiplications	Pupils arrive and are asked to enter the room Starter/ Bell work - **exercise**: simplifying ratios
10-20mins	Class work / teaching	Key objectives on board http: // m1maths.co.uk/tasks/librarv/loadlesson.asr2?title =ratio/ratiodividing pages 1 - 5
20-35 mins		Consolidation **exercise** questions displayed on notebook
35-40 mins		Selected pupils to demonstrate how they got answers.
40-50mins	Plenary - Mymaths pg 6 - 8	
50-55mins	Plenary : 3 questions from **exercise** 17 on calculating ratios Students leave	Pack away

Note : No judgements are being made about the quality of the lesson plan just the use of pedagogical terminology is being examined.

Appendix 22 – Pre Study Lesson Plan from Teacher D

Time	What is teacher doing?	What are the pupils doing?	My comment
10	Prepare for Learning (settling and recapping) Physical environment: Students will line up outside and then enter the room calmly standing behind their chairs. Hook or bell activity: Name the parts of the circle	Students are asked to see if they can in l0mins remember what any parts of a circle are called. They are to do this in a colour of their choice.	First learning episode to engage pupils is an activity similar to the one in the research lesson (pupils recalling prior learning)
5	**Agree Learning Outcomes** (use the learning objectives slide and make reference to Bloom's taxonomy to ensure appropriate levels of challenge) **Bronze:** recall the parts of a circle (level 5) **Silver :** discover the value of pi (level 5a) **Gold:** calculate the circumference using the radius or diameter (level 6b)	Pupils to record the learning objective and date ready to begin the lesson. Outcomes discussed and pupils decide which medal they would like to achieve by the end of the lesson.	
	Literacy Focus (use the half termly literacy focus or information from marking and assessment to inform the lessons specific literacy focus) **Literacy Focus** (whole school): Colons and Semi - Colons **Literacy Focus** (Specific): Use of key words through learning discussions taking place and key word bingo	Pupils are to mark off the key word on their bingo card when they hear it spoken by the teacher - they must be aware of the definition as if they call bingo they must provide some definitions.	

Time	What is teacher doing?	What are the pupils doing?	My comment
5	**Presenting New Information** to students (starter activity) Go over parts of a circle, did they miss any or get any wrong?	Pupils are to change the colour of the pen used for the starter. They are to change/edit any parts of the circle t hey had forgotten or got it wrong in the starter.	
10	**Active Learning** (students to process info and make sense of it. What are they doing with the new information to develop their knowledge and (understanding?) Finding pi activity involving measuring circular items	Students are given a variety of objects and have a table to fill in using the circumference and the diameter. They are to try and discover what pi is. Do they notice anything?	This is similar to a **task** in the research lesson where pupils are using objects (manipulatives) to discover learning.
15	**Demonstration of New Understanding** (take new learning and use in a different context. Students to show that they understand - not just repeating or recalling info). Working out the circumference of circles. **Extension activity** provided for those who are working well and understand task. **2nd extension activity**- semi circle perimeter? Teacher will circulate the room and move students onto the extension if necessary.	Students complete the example and then the circumference questions. Students move onto the extension- what if they don't have the diameter. Extension 2- semi circle extension. SF may need to become expert if all extension tasks are completed	The term **activity** in this lesson plan is similar to an **exercise** in the research. The learning episode appears to have the learning directed by the teacher with the progression to semi-circles rather than the pupils directing the learning.

Time	What is teacher doing?	What are the pupils doing?	My comment
10	**Review** (have learning outcomes been achieved? If so, how? If not, why not? Develop student ability to reflect upon their learning and how they can improve) AfL board question to answer as group	Pupils to review progress using questions from AfL board. Once finished they will put their book in either the red, yellow, or green basket dependent upon how much progress they think they have made.	

Appendix 23 – Post Study Lesson Plan from Teacher A

Lesson Plan: Bold is my emphasis – showing a consistent use of terminology with the research lesson.

Class : Year 9	Date : :July 16th
Lesson Aims	Use y=mx + c to find the equation of a line
Desired Learning Outcomes -	

Class : Year 9		Date :July 16th
Time	Pupils' activities	Teacher's inputs and activities
10	A card sort **activity** Pupils work in pairs so sort a set of cards into two groups those relating to straight line graphs and those relating to graphs such as bar graphs / pictograms line graphs. Also a third group of cards has key words / vocabulary these need sorting.	Explain the activity. Feedback will be taken from a number of pupil pairs. Misconceptions about graphs / charts which are not straight lines will be addressed
10	Main teaching / learning Powerpoint of how to construct a straight line graph from a given equation using a plotting table	Q and A Session against a powerpoint – 2 slides only
15	**Exercise** In pairs – pupils make up their own equation to plot.	Encourage some to look at a fractional coefficient for x Others to investigate a negative coefficient for x
20	**Skill** With a partner work the equations of the 4 lines Make up a question for another pair of pupils to solve.	Plot and join following the points (1, 5),(4, 4), 6, -3) (3, -2) You should have a parallelogram.
10	**Plenary – Task** Present a spreadsheet which plots y = 2x + 1 The spreadsheet has a second graph superimposed y = 2x + 5 **Challenge** Can the group change y=2x + 5 so that it 1 Crosses y = 2x + 1 (at any angle) 2 Crosses y = 2x + 1 (at a right angle)	This might be as a class or if the computers (laptops) are available then as a paired **task**

Appendix 24 – Post Study Lesson Plan from Teacher D

Lesson Plan: Bold is my emphasis – showing a consistent use of terminology with the research lesson.

Class : Year 8	Date : July 16th	
Time	Pupils' activities	Teacher's inputs and activities
0 to 5 mins	Pupils enter room and settle. Write date, title and learning objectives in book.	Ensure pupils enter room in a suitable manner with appropriate uniform. Hand books out, and make sure pupils have all materials they need for the lesson. Share learning objectives and outcomes, ask pupils to read each learning objective aloud.
5 to 15mins	**Starter Activity** What is area? Using the A3 sheet of paper, working with a partner write down all the things you know about area. Give a ten minute time limit to for the **activity**.	Ask pupils to work in pairs on the **activity.** Pupils asked to recall area facts. Circulate just to observe the answers
15 to 25mins	Main Teaching – **developing skills** QandA. What does the word area mean? Q.and A How do we calculate the area of a rectangle? **Skill** - Use the small 1cm by 1cm tiles to work the area of the big yellow rectangular piece of paper (7cm by 4 cm) QandA. How many little tiles are in one row? How many rows of little tiles are there? How many little tiles. Is there a nice easy way of doing the calculation? Discuss the meaning of area Explain that's area is always for a 2d shape. Ask pupils asked to write a sentence for an easy way to calculate the area of a rectangle.	

Time	Pupils' activities	Teacher's inputs and activities
25 to 35mins	**Exercise 1** Pupils to complete some simple – questions on calculating areas of rectangles (all integers) from the textbook. **Exercise 2** In pairs make up as many questions as you can. **Extension exercise** – How can we calculate the area of a triangle?	Circulate encourage pupils to think of non – integer lengths.
35 to 55 mins	**Plenary Task** The **task** is : The post office is designing a new stamp to celebrate our school's 50th birthday. A stamp measures 2 cm by 3cm – how many stamps will there be on a sheet of paper 42 cm by 30 cm (A3 paper) Ask pupils to make up a question of their own.	A3 paper available A number of tiles (stamp measuring 2 by 3) available for each pair Worksheet to record results. Feedback re answers. Collect sheets. HOMEWORK; hand out and explain homework. Due Thursday.

Appendix 25 – Questions devised by pupils
(In the Skills part of the lesson)

The original 4 questions (appendix 4) were

1. How many $\frac{1}{8}'$ s are there in $\frac{1}{4}$?
2. How many $\frac{1}{12}'$ s are there in $\frac{1}{3}$?
3. How many $\frac{1}{8}$'s are there in $\frac{1}{2}$?
4. How many $\frac{1}{32}$'s are there in $\frac{1}{8}$?

All 21 pupil pairings from both 7NR and 7AC answered these 4 questions correctly. They went to devise their own questions

Class	Questions Devised	Answered correctly	Incorrect solutions	Supplementary Questions from teacher
7AC	29	29	0	2
7NR	3	3	0	
Total	32	32	0	2

Examples of the types of questions devised by one pupil pair in 7AC, these are representative of both groups

How many 1/15's are there in a 1/3	How many 1/20's are there in a 1/4	How many 1/24's are there in a 1/3	How many 1/18's are there in a 1/6	How many 1/16's are there in a 1/4

The supplementary questions were both answered correctly by different pupil pairs.

How many $\frac{4}{6}'$ s are there in $\frac{2}{3}$?

How many $\frac{2}{9}'$ s are there in $\frac{1}{3}$?

Appendix 26 – Questions devised by pupils
(In the Exercise part of the lesson)

There were no set questions just the following example

$$\boxed{\frac{1}{2}} = \boxed{\frac{1}{4}} + \boxed{\frac{1}{4}}$$

All 21 pupil pairings from both 7NR and 7AC devised questions and every pair answered the vast majority of their questions correctly.

Class	Questions Devised	Answered correctly	Incorrect solutions
7AC	81	74	7
7NR	28	23	5
Total	109	97	12

Appendix 27 – Example Post lesson Pupil feedback

1. When you worked with the fraction tiles how did they help you understand the fractions?

They helped me understand because I could see them and help me understand it is better this way I think.

2 How easy did you find the making up your own 'What if Questions'?

I think it was a bit harder then answering one but me and my partner did make one.

3 How easy was the last task finding the answer to the concentrated quash problem?

It was easyer with the tiles but it was a little difficult.

4. Most of the lesson involved you working with another pupil. How did this help you understand fractions?

How many of the other smaller fractions fit in the bigger ones.

5. Which part of the lesson do you need to do more work on?

The last part with the concentrated squash questions and my own question.

Appendix 28 - Pupil feedback from 7AC

When you work with the fractions tiles how did they help you understand fractions	How easy did you find the making up of your own "What if questions"?	How easy was the last tasks finding the answer to the squash problem?	Most of the lesson involved you working with another pupil. How did this help you understand fractions?	Which part of the lesson do you need to do more work on?
The made understand how many lower fractions go in the bigger one	Easy	At the start it was complicated but Jason made me get it		
They helped me understand what goes into what		I was hard but after sir explained it again	She understood me than me so it was a help	
They helped me to understand because I was physically able to do it	I found it quite easy	I found it actually quite easy		
They helped me with the GCSE question [Wksheet 3]	Easy	It wasn't too hard	I listened to her ideas	Last question
They helped because it had a visual and I could see what when in the GCSE question [Wksheet 3]				
They helped me with the hard questions	I didn't do it but if I did I would be okay	Quite hard to be fair	She talked to me about it when I got stuck	The sorting out the tiles
They helped me with the hard question which made me revise how to look at fractions				
They showed me how many different fractions go into a whole which helped when doing the questions				

When you work with the fractions tiles how did they help you understand fractions	**How easy did you find the making up of your own "What if questions"?**	**How easy was the last tasks finding the answer to the squash problem?**	**Most of the lesson involved you working with another pupil. How did this help you understand fractions?**	**Which part of the lesson do you need to do more work on?**
They helped me understand how the denominator is x2 every time we halved the tile	Easy	Easy	We worked together to complete the questions	The squash problem
They helped me because it had a visual effect and you could see how many made a whole.				
It helped me because I could see the fractions so it was easier	It was a little bit easy			
Yes because you have the shape made ready				
Because I could actually see the fractions	Very			
Yes, because I could actually see the fractions				
Yes, It helped me because you can see the fractions. You can put them together too.				
Practical based get us active and doing it	Quite easy	Simple	Yes because I like teamwork	
It made it easier because I could visually see what I needed to work out.	I found it easy because I had the example questions to refer to	It was easy once I knew what I was doing	Becuase we both put our ideas together	

When you work with the fractions tiles how did they help you understand fractions	How easy did you find the making up of your own "What if questions"?	How easy was the last tasks finding the answer to the squash problem?	Most of the lesson involved you working with another pupil. How did this help you understand fractions?	Which part of the lesson do you need to do more work on?
They helped me understand fractions because they were in fractions	I found it pretty easy	It was a little hard	Because I helped her understand some questions and she helped me	The last task
They made me understand how many lower fractions go in the bigger fraction	Yes, easy	At the start it was frustrating but now it is easy	Because we have our own opinion	None because the teacher discussed and helped me and other pupils
They helped because I could use them and put them on the full piece [whole][of paper to see if I have covered half or something.	I found it very easy	I also found it quite easy		
They were a physical representation	Didn't do it	Very	It gave an extra mind to work the problem out.	None
	Didn't do it	It was hard and easy as we was given the hard sheet [sheet 3b]		None
They helped me understand halves of fractions	I found them quite easy to make up	It was actually quite easy	Because when I didn't understand something Courtney did.	
They helped me OK, but once I used the tiles once the I go on	Very easy	Easy	Very well because me and Amber are good friends	

Appendix 29 – Pupil Feedback from 7NR

When you work with the fractions tiles how did they help you understand fractions	How easy did you find the making up of your own "What if questions"?	How easy was the last tasks finding the answer to the squash problem?	Most of the lesson involved you working with another pupil. How did this help you understand fractions?	Which part of the lesson do you need to do more work on?
They helped because I could explain better	It was so easy to work out	It was easy because of the tiles	Things I did not [k][now, she helped me with	The last worksheet
Because we made it easier	A [???] bit hard	Didn't do it	Because we had someone to help	The solving questions
It helped me that I knew more fraction	It was a bit difficult	It was hard bit after it was good	Because we worked as a team and the tiles	The last question
It helped because I could experiment with them	I found it hard because I couldn't work it out	Easy because I go to add up all the numbers	It helped because Im better in paired groups	The ending question (What if)
Because it's easy	It was a bi hard	Didn't do it	Because we had someone to help	Solving questions
They helped me because I know how many is in a half and a quarter	It was hard	It was a bit hard	I worked with on my own	The squash part
We could use them to answer the questions	It was easy when we worked together	It was easy when it was explained to me	We could work together	
Being using objects physically	Quite easy and challenging	It was awesome really easy thinking ?? Maths	Not really I worked out all the questions	None I think I was good

When you work with the fractions tiles how did they help you understand fractions	How easy did you find the making up of your own "What if questions"?	How easy was the last tasks finding the answer to the squash problem?	Most of the lesson involved you working with another pupil. How did this help you understand fractions?	Which part of the lesson do you need to do more work on?
We found the questions a lot easier with the tiles	I didn't find it easy	I found it quite easy	I find it easy with other pupils than working on my own.	
Because it helped by getting me more confident		It was quite hard but easy when we got help	We worked as a team	More on fractions itself
They helped me understand fractions more and there easier to use	I didnt get time to make up my own but If I did find the questions easy	It was actually really fun, easy, interesting and challenging	This helped out very well with fractions	
They helped me very well with counting them up	it was harder than having to solve them	It was not easy but it was not hard	It helped a lot. We both used our knowledge to solve it.	Our own what if questions
They helped me understand because I could see them and help me understand it is better this way I think	I think it was a bit harder then answering one but me an my partner did make one	It was easier with the tiles but it was a little difficult	How many of the other smaller fractions fit in the bigger one	The last part with the concentrated squash and my own questions
They helped to show what they stood for	It was kinda hard due to I didn't think of any idea for it	Challenging	It helped	

When you work with the fractions tiles how did they help you understand fractions	How easy did you find the making up of your own "What if questions"?	How easy was the last tasks finding the answer to the squash problem?	Most of the lesson involved you working with another pupil. How did this help you understand fractions?	Which part of the lesson do you need to do more work on?
They helped by putting them together to make the right fraction	I didn't have time to make up my own but I did find he questions easy	It was easy working with my partner	This helped because she helped me and I did other things	I was good at everything
They helped me to see how many fractions were in a fraction	I thought it was easy	I thought it was easy	I worked on my own	Nothing

Appendix 30 - Video clips recording sheet for initial analysis

Recording Sheet Number _____ Lesson / Camera (1, 2, 3, 4, 5) ______

Pupil Pair _____

Video clip start time __________ Video clip end time __________

Part of the lesson (Activity, Exercise, Skill or Task) ___________

Time	Pupil	Dialogue	Action	Comment	Code*

In total there were over 400 of these forms which were then used to code the interactions to compile the summaries in appendices 33 and 34*.

The completion of this table happened in stages.

Stage 1: View the videos to complete columns 1 to 4 (Time, Pupil, Dialogue, Action) for all 400+ sequences. This gave an overview of the material and the sorts of interactions.

Stage 2: View the videos again to complete column 5 (Comments)

Stage 3: Devise the set of codes for the interactions appendices 33 and 34.

Stage 4: View the videos to complete column 6 (Codes)

Stage 5: View the videos again as check.

Appendix 31 – Example of a completed record sheet

Recording Sheet Number : **21**

Lesson / Camera (1, 2, 3, 4, 5) **: 7AC Camera 2**

Pupil Pair : **E**

Video clip start time **0 mins 5 secs**

Video clip end time **1 mins 35 secs**

Part of the lesson (Activity, Exercise, Skill or Task) Task – the squash problem

Time	Pupil	Dialogue	Action	Comment	Code*
0:05 – 0:09	1	There are 8 glasses in a litre. So we can divide a litre into 8 parts		Reading from the sheet	4 seconds only not counted
0:09 – 0:13	2	We can use this green card and if we put 8 together we have got the bottle of squash	Gets the card which represents the correct fraction		Just 4 seconds in length not counted.
0:13 - 0:16	1	But we also need 3 more.	Interpreting the question	Wrongly interpretin g the question 3 tiles for three quarters	3 seconds only – not counted.
0:16 - 0:26	2	No, we got to find out what three-quarters of 8 is. How many eighths are in three quarters?	Pupil 1 is collecting the tiles to work out what pupil 2 is saying and placing them on the desk – demoing the problem	Pupil 2 is using the language of division from the exercise and skills part of the lesson.	10 seconds in length interactions counted Code B Code E Code G

0:26 - 0:41	2	If we divide the three quarters in eighths we will get 6		The pair are performing the calculation on a white board – they do not use the manipulatives	15 seconds Code H
0:41 – 0:45	2	So one and three quarters would be 8 + 6 which is 14.	Pupil 1 is listening		Just 4 seconds in length not counted – it would have been code G
0:45 – 0:56	1	So is the answer 14.	Pupil is starting to write the answer on to the worksheet See below for the actual transcript Figure 1.		
0:56 – 1:08	At this point they clear the manipulatives away that they have been using				
1:08 – 1:29	1	Reads the next part of the worksheet out loud. He does this twice.	Pupil 2 is listening.	What if Question ... What if a bottle held 2 and a half litres of concentrate and the glasses needed one twelfth of a litre	11 seconds
1:29 – 1:35	2	It's like the last question but we need different one [tiles].	Pupil 1 gets the tiles which represent a twelfth.		
1:35 -	At this point the camera moves on to a different pupil pairing – however what the pupil pair produced is shown in figure 2 – and one might conjecturer they followed the same processes as above				

*Codes are in appendix 33 – and this is aggregated as part of appendix 33 - table 33.4

Appendix 32 – Example of a completed record sheet - activity
Recording Sheet Number **46**

Lesson / Camera (1, 2, 3, 4, 5) : **7AC Camera 1**

Pupil Pair : **A**

Video clip start time **0 mins 5 secs**

Video clip end time **1 mins 43 secs**

Time	Pupil	Dialogue	Action	Comment	Code*
0:05 – 0:12	1	Fractions have a top and a bottom number	Pupil 2 writing on the sheet		B
0:09 – 0:12	2	Numerator is the top number	Write Numerator is the top number on the worksheet		B
0:12 - 0:19		Pause		Neither pupil talking or saying anything audible on the video.	Nothing to code
0:16 - 0:26	1	Fractions are like percentages	Pupil l 2 writes fractions are really percentages		B
0:27 – 0.45	1	Pupil 1 turns around to speak to a different pair of pupils (4 seconds) This is not poor behaviour just healthy competition.			
0:46 – 1:10	Pupils talking about things not related to the activity				
1.11 – 1:26	1		Pupil writes denominator is the bottom number	There is no dialogue between the pupils.	

Time	Pupil	Dialogue	Action	Comment	Code*
1:26 -1:35	2	Did you add fractions in Mrs XXX class in school YYYY. Remember we had those charts on the wall and in our maths book we made them with pizza and cakes	Pupil 1 writes you can add fractions	They are talking about primary school	F
1:35 – 1:43	2	We also had that chart where different fractions were equal like ½ was 2/4	Pupil 1 writes ½ is 2/4	Recalling fraction facts	F
1:43 -	At this point the camera moves on to a different pupil pairing –				

*Codes are in appendix 34 – and this is aggregated as part of appendix 33 - table 33.1

Appendix 33 – Video Analysis of the lesson with 7AC.

The four tables indicate the number of pupil pairs (highlighted) engaging in the actions defined by A-H below which were longer than 5 seconds in length during each of the 4 learning episodes in the research lesson

Four cameras were used, three by participant teachers and one static at the front of the room. The tables show the length of time, chronologically from the beginning of the lesson for the 4 types of learning episodes being investigated. Eight categories of actions were defined, these being

Actions

A – Teacher Input – teaching / demonstrating / explaining
B – Pupil – Pupil Dialogue
C – Pupil Reasoning
D – Interventions
E – Pupils Using the manipulatives
F – Pupil – Teacher Dialogue
G – Pupil demonstrating understanding
H – Connecting learning (eg division of numbers with division of fractions)

The tables show how many pupils pairs were visible on the section of video (Pairs) and how many engage in these actions during each of the four learning episodes and when in the lesson these interactions occur. Where there are no frequencies this indicates that even though the video was observing pupil pairs they did not engage in any of the 8 actions.

Table 33.1

			Learning Episode - Activity							
Video	Length mm:ss	Pairs	A	B	C	D	E	F	G	H
Camera 1	0:01									
	3:00	4	1	3		1				
	1:38	1				2		2		
	11:01	5								
	5:55	3								
	8:45	3								
	2:24	1								
Camera 2	44:19	11	1	4	1	2	4	1	1	1
Camera 3	1:37	1	1	2				1		
	3:51	2							1	1
	17:06	7							2	
	16:18	7								
Camera 4	46.15	11	1	3	1	2	1	1	1	1
	5.24	1								

Table 33.2

Video	Length mm:ss	Pairs	Learning Episode - Skill							
			A	B	C	D	E	F	G	H
Camera 1	0:01									
	3:00	4								
	1:38	1								
	11:01	5	2	11			7			
	5:55	3								
	8:45	3								
	2:24	1								
Camera 2	44:19	11	1	2	2	3	2	1	1	2
Camera 3	1:37	1								
	3:51	2	1	2						
	17:06	7	2	2	2			2	2	2
	16:18	7								
Camera 4	46.15	11	5	2	1	4	3	4	3	2
	5.24	1								

Table 33.3

Video	Length mm:ss	Pairs	Learning Episode - Exercise							
			A	B	C	D	E	F	G	H
Camera 1	0:01									
	3:00	4								
	1:38	1								
	11:01	5	1	2				1	1	
	5:55	3	1		1	1	2			1
	8:45	3								
	2:24	1								
Camera 2	44:19	11	3	2			2	2	2	3
Camera 3	1:37	1								
	3:51	2								
	17:06	7	3	1	1	1	2	1	1	2
	16:18	7	1					1	1	1
Camera 4	46.15	11	2	2	1	1	2	1		
	5.24	1								

Table 33.4

Video	Length mm:ss	Pairs	Learning Episode - Task							
			A	B	C	D	E	F	G	H
Camera 1	0:01									
	3:00	4								
	1:38	1								
	11:01	5								
	5:55	3								
	8:45	3	1	1	1					
	2:24	1	1	2	1	1	2			2
Camera 2	44:19	11	4	1	1	1	2	1	1	4
Camera 3	1:37	1								
	3:51	2								
	17:06	7								
	16:18	7	4	1	1	2	1	1	2	1
Camera 4	46.15	11	4	2	2	3	1	1	2	3
	5.24	1	2	1		1	1	1	1	2

Appendix 34 – Video Analysis of the lesson with 7NR.

The four tables indicate the number of pupil pairs (highlighted) engaging in the actions defined by A-H below which were longer than 5 seconds in length during each of the 4 learning episodes in the research lesson

Five cameras were used, four by participant teachers and one static at the front of the room. The tables show the length of time, chronologically from the beginning of the lesson for the 4 types of learning episodes being investigated. Eight categories of actions were defined, these being

Actions

A – Teacher Input – teaching / demonstrating / explaining
B – Pupil – Pupil Dialogue
C – Pupil Reasoning
D – Interventions
E – Pupils Using the manipulatives
F – Pupil – Teacher Dialogue
G – Pupil demonstrating understanding
H – Connecting learning (eg division of numbers with division of fractions)

The tables show how many pupils pairs were visible on the section of video (Pairs) and how many engage in these actions during each of the four learning episodes and when in the lesson these interactions occur. Where there are no frequencies this indicates that even though the video was observing pupil pairs they did not engage in any of the 8 actions.

Table 34.1

Video	Length mm:ss	Pairs Observed	**Learning Episode - Activities**							
			A	B	C	D	E	F	G	H
Camera 1	1:33	1	1		2	1	1		1	1
	0:23	1	1			1				
	9:14	4	2	1	1	1				1
	1:27	1	1			1	1	1	1	
	0:49	1								
	6:29	2								
	2:52	1								
	0:01	-								
	0:41	1								
	1:02	1								
	0;55	1								
	0:57	1								
	1:35	1								
Camera 2	16:53	5	3	1		1	1		3	1
	0:30	1								
	0:48	1								
	3:04	1								
	1:44	1								
	0:35	1								
	1:42	1								
	4:16	2								
	2:02	1								
Camera 3	13:31	6	5		5	3	3	1		
	5:36	3								
	12:36	4								
	2:35	1								
	2:59	1								
	2:52	1								
Camera 4	43:21	12	8	3	1		1	1	2	1
Camera 5	44.20	10	9		2	2			2	1
	12.58	4								

Table 34.2

Video	Length mm:ss	Pairs Observed	Learning Episode - Skill A	B	C	D	E	F	G	H
Camera 1	1:33	1								
	0:23	1								
	9:14	4								
	1:27	1	1	1			1			
	0:49	1	1			1		2	1	
	6:29	2								
	2:52	1								
	0:01	-								
	0:41	1								
	1:02	1								
	0;55	1								
	0:57	1								
	1:35	1								
Camera 2	16:53	5	2		1	1	1		1	3
	0:30	1								
	0:48	1								
	3:04	1								
	1:44	1								
	0:35	1								
	1:42	1								
	4:16	2								
	2:02	1								
Camera 3	13:31	6	3	3	2	2		1	1	
	5:36	3								
	12:36	4								
	2:35	1								
	2:59	1								
	2:52	1								
Camera 4	43:21	12	3		2	4	2	1		1
Camera 5	44.20	10	4	2		2	4		2	
	12.58	4								

Table 34.3

			Learning Episode - Exercise							
Video	Length mm:ss	Pairs observed	A	B	C	D	E	F	G	H
Camera 1	1:33	1								
	0:23	1								
	9:14	4								
	1:27	1								
	0:49	1	1	2	1	1			1	1
	6:29	2	1				1			
	2:52	1								
	0:01	-								
	0:41	1								
	1:02	1								
	0;55	1								
	0:57	1								
	1:35	1								
Camera 2	16:53	5	3	1	1	1				
	0:30	1	1			1		1	1	
	0:48	1	1				2			
	3:04	1								
	1:44	1								
	0:35	1								
	1:42	1								
	4:16	2	1		1		1	1	1	
	2:02	1				1			1	
Camera 3	13:31	6		3	3					3
	5:36	3	1		1		1	1		
	12:36	4			2			2	1	
	2:35	1								
	2:59	1								
	2:52	1								
Camera 4	43:21	12	5		1		3	2	1	
Camera 5	44.20	10	3	1		2	1	4	1	1
	12.58	4	2		1	1	1	2	1	

Table 34.4

Video	Length mm:ss	Pairs observed	Learning Episode - Task							
			A	B	C	D	E	F	G	H
Camera 1	1:33	1								
	0:23	1								
	9:14	4								
	1:27	1								
	0:49	1								
	6:29	2								
	2:52	1	1	1	1					
	0:01	-								
	0:41	1		1	1			1	1	2
	1:02	1				1	2	1		
	0;55	1			1				1	1
	0:57	1				1	1			
	1:35	1	1				1	1	1	
Camera 2	16:53	5								
	0:30	1								
	0:48	1	1		2	2	1	1	2	
	3:04	1		1			2	2		
	1:44	1								
	0:35	1								
	1:42	1								
	4:16	2	1			1	1	1		1
	2:02	1			1	1			1	1
Camera 3	13:31	6								
	5:36	3	3	1	1					
	12:36	4					1	1		
	2:35	1			2	2			2	1
	2:59	1				1	2	3	1	1
	2:52	1				1	1	1	2	2
Camera 4	43:21	12	4		3	2	1		4	4
Camera 5	44.20	10	2	4	3	2	1	2	3	1
	12.58	4	2	3	1	2		1	1	1

Appendix 35 – Definitions of Pedagogical Terminology.

An Exercise: is an aspect of pedagogy designed by a teacher to encourage learning and is seen as an extension to drill and practice type questions to gain an understanding of a mathematical concept. Additionally the newly acquired piece of learning is explored through a limited number of teacher prescribed questions and importantly extended by an additional set of learner generated questions.

An Activity: is an aspect of pedagogy designed by a teacher where its main feature is that no new learning is presented, however, new learning may naturally occur. The learning aspect is designed for groups of learners to recall previously acquired knowledge, or to engage the learner, or as a hook into the next part of the learning.

A Skill: is an aspect of pedagogy designed by a teacher where a new piece of mathematical knowledge is presented and the learner is required to complete a number of prescribed questions to practise and perfect the new learning, an example would be the division of two fractions.

A Task: is an aspect of pedagogy designed by a teacher to prompt the application of newly acquired mathematical knowledge or learning to either a real-life problem (purpose) or a contrived but messy situation and has a usefulness (utility). A task is a piece of learning which requires some exposition and explanation from the teacher and often follows a prescribed pathway but has the opportunity for pupils to define their own problems and promotes mathematical discussions. The mathematical knowledge required for the task will have been acquired and developed in the exercises, activities and skills part of the lesson.

Whilst trying to frame these four pedagogical terms so that there uniqueness is evident it is obvious that this is both difficult and in some respects counterproductive. The definitions do aim to give the distinct features of each term whilst recognising that learning and new mathematical knowledge can spontaneously occur irrespective of the pedagogical aspect selected by the teacher.

Appendix 36 - Ethical Forms and information to parents.

The information and consent forms were designed by me but the school governors insisted they were sent by a member of staff on school headed noted paper.

.

Date

We have worked closely with Wolverhampton University over the past few years on projects to help with the training of teachers and the teaching of Mathematics. We have been asked if we can help with a research project about how pupils learn Mathematics. This will benefit the University and XXXX and in turn should help to improve some aspects of our future teaching. As part of the project students will work in their normal class in small groups with a member of staff from Wolverhampton University for about 45 minutes, and their responses will be filmed to be analysed later. The attached form outlines the details of the filming. This will take place in school the week beginning the 15^{th} July. The process *will a/so* be observed by a number of XXXX teachers. The results of the research will be shared with the school at a later date.

If you are happy that your child takes part in this project, can you please sign and return the attached form to Mrs A.B by Wednesday 11th July.

Thank you in advance for your support.

Yours faithfully

Mrs A.B Head of Mathematics

CONSENT TO PHOTOGRAPH, FILM OR VIDEOTAPE A STUDENT FOR EDUCATIONAL USE

Name of Student: Class:

I ____________________ (Parent or Guardian Name), hereby consent to the participation in the taking of videotapes of my son/daughter, and his/her school-related work by XXXXX *Cot* E High School. The video material will only be used by the school and the University of Wolverhampton to investigate the best ways of pupils learning Mathematics. All the material will be stored securely and no video or images will be available to the public or be published anywhere. I also hereby release XXXX *Cot* E High School and Wolverhampton University and employees from all claims, demands and liabilities whatsoever in connection with the above.

An identical set of forms were designed and issued for the participating teachers.

Appendix 37 - Research Ethical Approval

UNIVERSITY OF WOLVERHAMPTON

Request for Ethical Approval

<u>Section 1</u> – to be completed by the researcher

Full name	Michael Rickhuss
Module number and title (student researchers only)	PhD Thesis
Research Proposal title	Exploring the influence of task design on the teaching and learning process and outcomes in secondary school mathematics.
Brief outline of proposal	This ethical approval request builds on the already agreed part 'A' ethical approval moving from a survey using questionnaires of professionals to researching with children and their teacher. A suite of tasks (relating to the teaching of fractions algorithms) have been designed to exemplify principles of task design. It is proposed that these be piloted in two classrooms in a single school. The interactions of the students will be audio/video taped. Semi-structured interviews will also be carried out (only audio-recorded) with the class teachers and their responses analysed.
Level of research, e.g. staff, undergraduate, postgraduate, master's (award related), MPhil, PhD	Staff – PhD
Please outline the methodology that would be implemented in the course of this research.	Groups of pupils working on the mathematical tasks will be audio/video – recorded and the results transcribed. The rationale for using video as a means of data capture is that non-verbal communication whilst pupils are working with tasks and mathematical objects (fractions) will indicate a level of additional understanding in addition to pupil dialogue. Written voluntary informed consent (see additional example documents) from the schools, teachers, parents and pupils for the audio and video will be sought before the research commences. All parties will have the right to withdraw from the research for any or no reason at any stage without question. The teachers of the two classes will be interviewed to explore

	their views of their children's learning. All video and audio material / images will be used solely for the research and will be erased once the research is completed. None of the video / audio material will be shared, except with the research supervisors, and only for validation purposes. The research report will be available for the participants after the study is completed. All data will be anonymous.
Please indicate the ethical issues that have been considered and how these will be addressed.	The anonymity of the participants is considered important and as such any identifying personal data will not be shared (see attached document for pupils). Data relating to gender and geographical location of individuals will be coded for comparisons.
Please indicate any issues that may arise relating to diversity and equality whilst undertaking this research and how you will manage these.	The research will be sensitive to issues relating to diversity and equality and the experiences of all participants (pupils, teachers and parents) and the impact on their beliefs and professional pedagogy. The research will be mindful of pupils existing knowledge and understanding and seek to minimise the potential for distress or conflicts in understanding.

Please answer the following questions by deleting the inappropriate response:

1. Will your research project involve young people under the age of 18? Yes

If yes, do you have an Enhanced Disclosure Certificate from the Criminal Records Bureau? Yes

2. Will your research project involve vulnerable adults? No

3. For which category of proposal are you applying for ethical approval? B

Confirmation of ethical approval

<u>Section 2</u> – to be completed as indicated, by module leader, supervisor and/or chair of ethics sub-committee

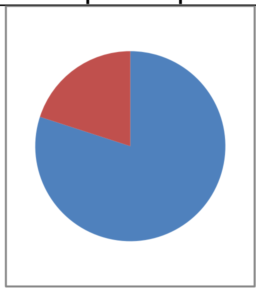
On behalf of members of staff <u>and</u> students

I confirm that the proposal for research being made by above student/member of staff is a category B proposal and that s/he may now continue with the proposed research activity:

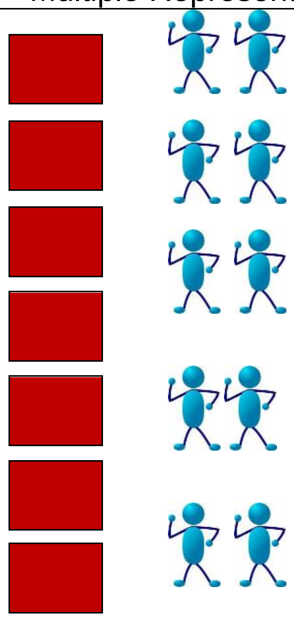
Signed	[Signature redacted]
Name of chair of ethics sub-committee	Dr L M DEVLIN
Any conditions attached to this ethical approval request.	**Approved**
Date	10.07.12

Appendix 38 – Five visualisations of a fraction.
Based on the work of Moseley (2005)

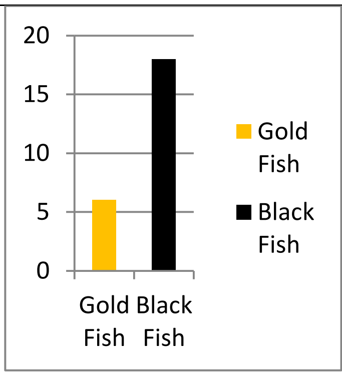
A Fraction as Part –to-Whole Sub-construct

Single Representation	Multiple Representation	Formal
A jar holds 50 ml of orange when full. The jar has 40 ml of orange left in it after Mike uses some. What fraction of the jar has orange left in it?		$\frac{4}{5}$

A Fraction as a Decimal Sub-construct

Single Representation	Multiple Representation	Formal
7 bars of chocolate need to be divided between 10 pupils. What fraction of the bar of chocolate will each pupil receive?		0.7

A Fraction as a Ratio Sub-construct

Single Representation	Multiple Representation	Formal
In a pond there are 6 goldfish and 18 black fish. What is the relationship of gold fish to black fish?		$\frac{1}{3}$

A Fraction as a Division Sub-construct

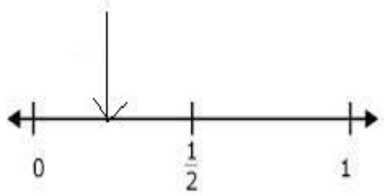
Single Representation	Multiple Representation	Formal
A piece of string is 3 metres long. It needs to be cut into 4 equal lengths. What is the length of each of the 4 pieces of string?	4m	$\frac{3}{4}$

A Fraction as an Operator Sub-construct

Single Representation	Multiple Representation	Formal
Work out $\frac{1}{2}$ of £36		$\frac{1}{2}$

A Fraction as a measure of discrete or continuous quantities Construct

Fractions as points on a number line

Single Representation	Multiple Representation	Formal
If 10 metres of fencing are erected out of a total of 40 metres, what fraction of the total fence is completed?		$\frac{1}{4}$

Appendix 39 – Categories of Questions Kvale (1996)

Example questions to prompt conceptual understanding and learning.

Introducing questions: 'Why did you...?' or 'Can you tell me about...?' Through these questions you introduce the topic.

Follow up questions: Through these you can elaborate on their initial answer. Questions may include: 'What did you mean...?' or 'Can you give more detail...?'

Probing questions: You can employ direct questioning to follow up what has been said and to get more detail. 'Do you have any examples?' or 'Could you say more about...?'

Specifying questions: Such as 'What happened when you said that?' or 'What did he say next?'

Direct questions: Questions with a yes or no answer are direct questions. You might want to leave these questions until the end so you don't lead the interviewee to answer a certain way.

Indirect questions: You can ask these to get the interviewee's true opinion.

Structuring questions: These move the interview on to the next subject. For example, 'Moving on to...'

Silence: Through pauses you can suggest to the interviewee that you want them to answer the question!
Interpreting questions: 'Do you mean that...?' or 'Is it correct that...?'

Appendix 40 – Skills Practice – Fractions

11. 20 min of 1 hour
12. 6 months of 1 year
13. 30° of 90°
14. 6 hours of 1 day
15. 2 days of 1 week
16. 120° of 180°
17. 18 hours of 1 day
18. 20 hours of 1 day
19. 36 min of 1 hour
20. 146 days of 1 year

Addition of fractions

In order to add fractions, we must give them a **common denominator**.

EXAMPLE 1 Add $\frac{2}{3} + \frac{5}{6} + \frac{7}{12}$

The LCM of the denominators is 12

$$\frac{2}{3} + \frac{5}{6} + \frac{8}{12} = \frac{8}{12} + \frac{10}{12} + \frac{7}{12}$$

This way looks neater → $= \frac{8 + 10 + 7}{12}$

$$= \frac{25}{12}$$

$$= 2\frac{1}{12}$$

EXAMPLE 2 Add $\frac{3}{5} + \frac{4}{15} + \frac{11}{20}$

LCM of 5, 15 and 20 is 60

$$\frac{3}{5} + \frac{4}{15} + \frac{11}{20} = \frac{36 + 16 + 33}{60}$$

$$= \frac{85}{60} = 1\frac{\cancel{25}^{5}}{\cancel{60}_{12}}$$

$$= 1\frac{5}{12}$$

Exercise 35

1. $\frac{1}{2}+\frac{1}{4}$
2. $\frac{1}{2}+\frac{1}{3}$
3. $\frac{1}{2}+\frac{1}{5}$

44

4. $\frac{1}{3}+\frac{1}{4}$
5. $\frac{1}{3}+\frac{1}{5}$
6. $\frac{1}{4}+\frac{1}{5}$
7. $\frac{2}{3}+\frac{1}{2}$
8. $\frac{2}{3}+\frac{1}{4}$
9. $\frac{2}{3}+\frac{2}{5}$
10. $\frac{1}{3}+\frac{3}{4}$
11. $\frac{2}{3}+\frac{3}{4}$
12. $\frac{1}{3}+\frac{2}{5}$
13. $\frac{1}{3}+\frac{3}{5}$
14. $\frac{2}{3}+\frac{4}{5}$
15. $\frac{1}{2}+\frac{3}{4}$
16. $\frac{1}{2}+\frac{2}{5}$
17. $\frac{1}{2}+\frac{1}{8}$
18. $\frac{1}{4}+\frac{1}{8}$
19. $\frac{1}{3}+\frac{1}{8}$
20. $\frac{1}{5}+\frac{1}{8}$
21. $\frac{2}{3}+\frac{3}{8}$
22. $\frac{3}{5}+\frac{5}{8}$
23. $\frac{3}{4}+\frac{3}{5}$
24. $\frac{2}{5}+\frac{7}{8}$
25. $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$
26. $\frac{1}{2}+\frac{2}{3}+\frac{1}{4}$
27. $\frac{1}{2}+\frac{1}{3}+\frac{3}{4}$
28. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}$
29. $\frac{1}{3}+\frac{1}{4}+\frac{1}{6}$
30. $\frac{2}{3}+\frac{1}{4}+\frac{1}{6}$
31. $\frac{1}{3}+\frac{3}{4}+\frac{1}{6}$
32. $\frac{1}{3}+\frac{1}{4}+\frac{5}{6}$
33. $\frac{2}{3}+\frac{1}{4}+\frac{5}{6}$
34. $\frac{2}{3}+\frac{3}{4}+\frac{5}{6}$
35. $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$
36. $\frac{1}{4}+\frac{3}{8}+\frac{3}{16}$
37. $\frac{3}{4}+\frac{1}{8}+\frac{5}{16}$
38. $\frac{3}{4}+\frac{3}{8}+\frac{3}{16}$
39. $\frac{3}{4}+\frac{5}{8}+\frac{7}{16}$
40. $\frac{2}{3}+\frac{3}{4}+\frac{5}{9}$
41. $\frac{1}{3}+\frac{1}{4}+\frac{8}{9}$
42. $\frac{3}{8}+\frac{1}{6}+\frac{5}{12}$
43. $\frac{5}{8}+\frac{5}{6}+\frac{7}{12}$
44. $\frac{2}{3}+\frac{5}{9}+\frac{7}{12}$
45. $\frac{1}{3}+\frac{5}{6}+\frac{3}{8}$
46. $\frac{7}{11}+\frac{9}{22}+\frac{13}{33}$
47. $\frac{2}{5}+\frac{3}{4}+\frac{7}{15}$
48. $\frac{11}{15}+\frac{7}{20}+\frac{5}{6}$

45

Taken from Fox, R.W. (1981) Basic Skills in Mathematics Book 2. Pitman Press. London pp. 44-45

Appendix 41 – Skills Practice Algebra

Exercise 6.8: Factorisation

Factorise the following expressions:

1. $4x+6y$
2. $10x+15y$
3. $11y+22z$
4. $24z-16x$
5. $9x-6y$
6. $5xy+7x$
7. $6yz+11y$
8. $3xy-5x$
9. $4xy-9x$
10. $20w-15wx$
11. $3x+9y+6z$
12. $21x+14y+28z$
13. $16xy+24x$
14. $10xy-15y$
15. $15xy-10yz$
16. $12xz+4xy$
17. $9xy-12yz$
18. $24xy-16xyz$
19. $18xyz+3wx$
20. $8ab+12abc$
21. $21x-7y$
22. $21x-7$
23. $8x-4y$
24. $8x-4$
25. $35y+7x$
26. $35y+7$
27. $3xy+2x$
28. $3xy+x$
29. $6xy-3yz$
30. $6xy-3y$
31. $5x^2+2x$
32. $3x^2+9xy$
33. $4x^2+8xy+24x$
34. $7x^2y-2x^2$
35. $8x^2y-2xz$
36. $8x^2y-2xy$
37. $16x^2y-12y^2$
38. $6a^2+9a+12ab$
39. $3a^3-9a^2$
40. a^3+2a^2-5a
41. $10a^3+5a^2-40a$
42. $10a^3+5a-25$
43. $2x^2+x$
44. $3x^3+2x^2+x$
45. $5x^2-x$
46. $2x^4+6x^2-8x$
47. $5xy+7x^3y-4x^2y^2$
48. $6xy+9x^2y-3xy^2$
49. $2(3+y)+x(3+y)$
50. $4(x+3)+y(x+3)$
51. $4(x+3)-y(x+3)$
52. $x(2y+1)-4(2y+1)$
53. $3(x-2)-x(x-2)$
54. $x(x+2y)+4(2y+x)$
55. $3(x+y)+2x(x+y)+4y(x+y)$
56. $6+2x+3y+xy$
57. $6-2x+3y-xy$
58. $6+2x-3y-xy$
59. $6-2x-3y+xy$
60. $x^2+3x+xy+3y$

Exercise 6.9: Bracket × bracket expansion

Expand and simplify the following:

1. $(x+2)(x+5)$
2. $(x+3)(x+2)$
3. $(x+4)(x+1)$
4. $(x+3)(x+7)$
5. $(x+3)(x+4)$
6. $(x+1)(x+6)$
7. $(x+2)(x+6)$
8. $(x+5))x+4)$
9. $(x+8)(x-2)$
10. $(x+5)(x-1)$
11. $(x+7)(x-3)$
12. $(x+2)(x-3)$
13. $(x+5)(x-7)$
14. $(x-3)(x+4)$
15. $(x-2)(x+5)$
16. $(x-7)(x+10)$
17. $(x-5)(x+1)$
18. $(x-9)(x+2)$
19. $(x-4)(x-3)$
20. $(x-1)(x-6)$
21. $(x-5)(x-3)$
22. $(x-2)(x-4)$
23. $(x-1)(x-3)$
24. $(x-9)(x-3)$
25. $(x+7)(x-2)$
26. $(2x+3)(x+2)$
27. $(3x+2)(x+3)$
28. $(4x+1)(2x-3)$
29. $(3x+2)(2x-1)$
30. $(3x-2)(2x+5)$
31. $(6x-3)(x+1)$
32. $(2x-5)(x+4)$
33. $(2x-5)(2x+7)$
34. $(3x-2)(2x-3)$
35. $(5x-1)(x-5)$
36. $(x+2)(x-2)$
37. $(x+3)(x-3)$
38. $(x-4)(x+4)$
39. $(x-5)(x+5)$
40. $(2x+3)(2x-3)$
41. $(3x-2)(3x+2)$
42. $(5x+y)(5x-y)$
43. $(x+4)^2$
44. $(x+3)^2$
45. $(x-2)^2$
46. $(x-7)^2$
47. $(2x+3)^2$
48. $(2x-3)^2$
49. $(x+y)^2$
50. $(2x-3y)^2$
51. $(x+3)^3$
52. $(x-4)^3$
53. $(2x+1)^3$
54. $(x+2y)^3$
55. $(x+3)(x^2+2x+3)$
56. $(x+1)(x^2+x+2)$
57. $(2x+3)(x^2-3x+1)$
58. $(3x-2)(x^2+2x-5)$
59. $(x+3)^4$
60. $(x-2)^4$

Exercise 6.10: Factorising trinomials; difference of two squares

Factorise the following expressions:

1. x^2+5x+6
2. $x^2+7x+12$
3. $x^2+9x+20$
4. $x^2+8x+15$
5. x^2+3x+2
6. x^2+4x+3
7. x^2+6x+8
8. $x^2+7x+10$
9. $x^2+8x+12$
10. $x^2+13x+12$
11. $x^2+14x+24$
12. $x^2+10x+24$
13. $x^2+11x+24$
14. $x^2+25x+24$
15. x^2+7x+6
16. $x^2+9x+18$
17. $x^2+11x+18$
18. x^2+4x+4
19. $x^2+8x+16$
20. $x^2+14x+49$
21. $x^2+4x-12$
22. x^2+x-12
23. $x^2+11x-12$
24. $x^2+4x-21$
25. $x^2-2x-15$
26. x^2+x-6
27. x^2-3x-4

Sadler, A. J. (1984) Pick and Choose – A Resource Bank of questions for CSE / ‘O’ level. Arnold London

Appendix 42 – Opposite Corners

Specimen Coursework task Syllabus 1385 Edexcel – 1998 (Lower and middle ability).

For Ainley and Pratt (2006) this mathematical task might lack purpose and utility but for Swann (2006) it certainly is a rich environment in which to explore mathematics from a number of differing avenues and provides opportunities to link topics together.

The diagram shows a 100 square. A rectangle has been shaded on the 100 square

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
14	42	43	44	45	64	47	48	49	50
51	52	53	54	55	65	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	37	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	95	95	96	97	98	99	100

The numbers in the opposite corners of the shaded rectangle are

54 and 66 and **64 and 56**

The products of the numbers in these opposite corners are

54 x 66 = 3564

64 x 56 = 3584

The difference between the products is

3584 – 3564 = 20

Investigate the different between the products of the numbers in the opposite corners of nay rectangles that can be drawn on a 100 square.

Appendix 43 – Patterns with Fractions

Specimen Coursework task Syllabus 1385 Edexcel – 2000 (Aimed at middle and higher attaining pupils)

PATTERNS WITH FRACTIONS

Consider the sequence of fractions

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$$

The **differences** between **consecutive fractions** are:

$$\frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}$$

The **next differences** between **consecutive fractions** are:

$$\frac{1}{12}, \frac{1}{30}$$

Investigate the difference patterns for this and / or other families of fractions.

Appendix 44 – Noughts and Crosses

Investigation Task taken from New York Cop Bell, Brown and Buckley (1989) – (Designed for pupils of all attainments above 40 percentile aged 13 years)

NOUGHTS AND CROSSES

O	x	x
O	x	O
X	x	O

O	O	x
O	x	O
x	x	x

x	O	O
x	x	O
x	O	x

The diagrams above show the winning lines for x in a game of noughts and crosses.

1. How many different winning lines are there altogether on a 3 by 3 grid

Noughts and crosses can also be played on a 4 by 4 grid. The diagram below shows a winning line for O. There must be 4 in a line.

O	O	O	O
O	x	x	x
x	x	x	O
x	O	x	O

2. How many winning ways are there on a 4 by 4 grid?

3. How many winning ways are there on a 2 by 2 grid?

4. Copy and complete the table

Size of Grid	Number of winning ways
2 by 2	
3 by 3	
4 by 4	
5 by 5	
6 by 6	

5. How many winning ways are there on a 20 by 20 grid?

6. How many winning ways are there on an n by n grid?

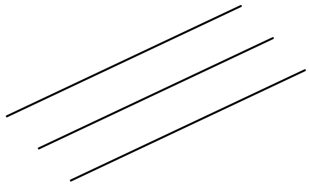
Appendix 45 – Crossed Lines

Investigation Task taken from New York Cop (1989) – (Designed for pupils of all attainments aged 13 years)

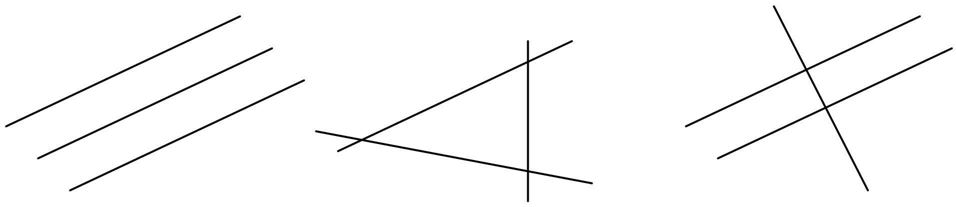
Crossed Lines

If you draw some straight lines on
Paper you can draw lines them so that

They do or do not cross.



Three straight lines can be drawn so that the lines never cross each other. Or so that one line crosses the other two.



Or so that all three line cross each other.

7. Lines which do not cross however far they are extended have a special name. What is it?
8. What is the maximum number of crossing points with five lines?
9. Copy and complete this table

Number of lines	1	2	3	4	5	6	7	8
Maximum number of crossing points								

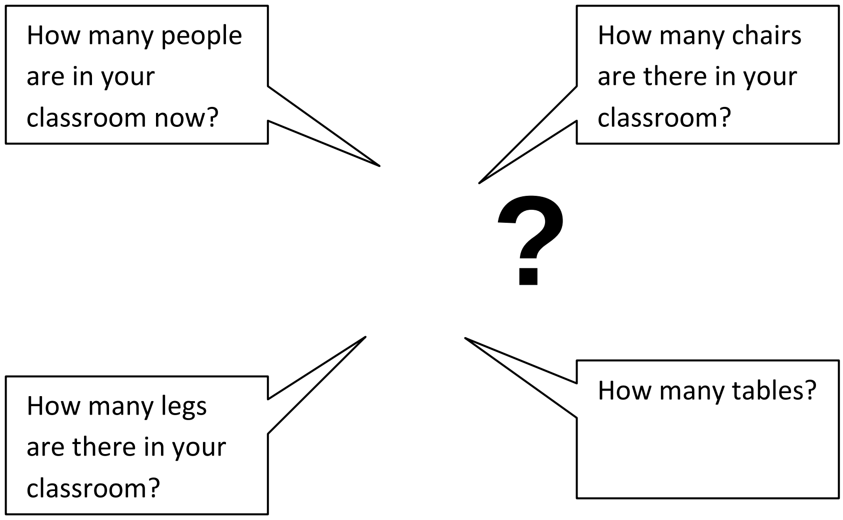
10. If ten lines were drawn so that they all crossed each other, how many crossing points would there be?
11. Now describe this pattern and write down a rule to give the next number in the sequence, and explain why it works.
12. Find out the name of the sequence of numbers in the crossing points row.
13. If n lines were drawn so that they all crossed each other, what would be the maximum number of crossing points?

Appendix 46 – How Big is your Classroom?

Rich Task taken from Rich Tasks 1 (2011) – (Designed for pupils of all attainments of all ages)

HOW BIG IS YOUR CLASSROOM?

You could answer this question in different ways
One way is to find how much will fit in your classroom.



How big is the classroom floor?

The classroom floor could be measured in many different ways. Here are some ideas.

1. How many sheets of newspaper would you need to cover the classroom floor completely?
2. How many tables could you fit in your classroom? (Each table has its legs in contact with the floor.)
3. How many people could stand on the floor of your classroom at the same time? Is this with or without furniture?
4. How many people could lie on the floor of your classroom at the same time?
5. Make a square with sides of length exactly one metre. (You could make it by sticking sheets of newspaper.)

How many of these squares would you need to cover the floor of your classroom? (You are allowed to cut the square to make them fit better.)

Appendix 47 – Three-Digit Fractions

Rich Task taken from Rich Tasks 1 (2011) – (Designed for all abilities all ages 9 - 90)

THREE - DIGIT FRACTIONS

The digits 1, 2 and 3 can be used to make several different three-digit fractions.

Here are some of them $\frac{1}{23}$ $\frac{2}{31}$ $\frac{3}{12}$ $\frac{21}{3}$ $\frac{13}{2}$

One of these fractions is a whole number. This is the fraction $\frac{21}{3}$

The three-digit fraction $\frac{12}{3}$ is also a whole number.

Making whole numbers

1. Use the digits 2, 3 and 4. Use the digits only once. Write down all the three-digit fractions you can find. Which of the fractions are whole numbers?
2. Now use the digits 3, 4 and 5. Find as many whole numbers as you can by making three-digit fractions.

It's impossible

1. If you use the digits 2, 3 and 5 you cannot form a three-digit fraction which is a whole number. Try to explain why.
2. Find another 2 sets of three digits which you cannot use to make a three-digit fraction which is a whole number.

Making a half

$\frac{7}{14}$ is a three-digit fraction which is equal to a half.

1. Find some other three-digit fractions which are equal to $\frac{1}{2}$

   Find all the three-digit fractions which are equal to $\frac{1}{2}$

   Explain how you know you have found them all.

2. Find some other three-digit fractions which are equal to $\frac{1}{3}$ and $\frac{1}{4}$ and ......

Something and a half

1. There is only one three-digit fraction which is equal to $1\frac{1}{2}$.

   What is it?

2. Find all the three-digit fraction which are equal to $2\frac{1}{2}$.

Smallest and largest

1. Make the smallest possible three-digit fraction with 2, 3 and 4.
2. Make the largest possible three-digit fraction with 2, 3 and 4.
3. Explain how you know the fractions you have made are the smallest and largest possible.
4. Answer question 1 using a different set of three digits.
5. What is the smallest fraction you can make if you are allowed to use any three digits?
6. What is the largest fraction you can make if you are allowed to use any three digits?
7. What is the largest number that can be made as two different three-digit fractions? It does not have to be a whole number?

Digits the same

In these three- digit fractions two digits are the same $\frac{3}{13}$ $\frac{7}{70}$

1. How many three-digit fractions can you make with 3, 3, 4?

2. Is there a fraction with two digits the same which is equal to

   $\frac{1}{2}$ or $\frac{1}{3}$ or ......

3. What happens if all three digits are the same?

How many three-digit fractions are there?

This is a challenging problem that will take some time to solve.

Six-digit sums?

Look at this fraction addition. It uses six different digits.

$$\frac{1}{6}+\frac{3}{9}=\frac{4}{8}$$

Find some other additions like this using six different digits.

Now find multiplications like this using six different digits.

$$\frac{4}{8}x\frac{3}{9}=\frac{1}{6}$$

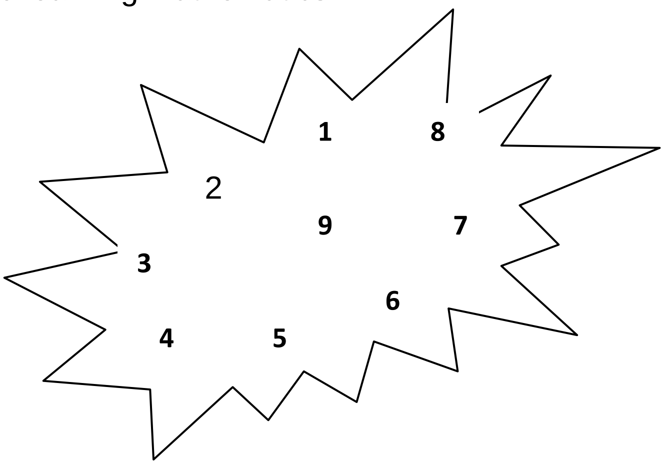
Appendix 48 – Two Example Tasks

This appendix shows two example tasks often seen in mathematics classroom.

Problem A – this is a standard task which develops systematic logical investigation and is of the type that most mathematics teachers would recognise as "a real world task" 

Suppose the post office only sells 3p and 5p stamps.
What different amounts of postage can made?
Which amounts of postage cannot be made using only 3p and 5p stamps?

Problem B – this is an investigational task which starts from what appears to be a fairly closed stance but has the potential for pupils to investigate and pursue a number of avenues and can foster creative approaches to learning mathematics



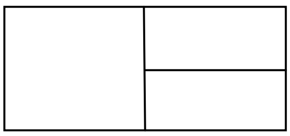
Instructions

A. Select any three digits from the cloud.
B. Form the six two-digit numbers that can be made.
C. Add the initial three digits
D. Total the six two-digit numbers
E. Divide these two answers.

Investigate.

Appendix 49 – Fraction Misconceptions

The table show eight of the most common misconceptions that pupils demonstrate when learning about fractions.

1.	The bigger the denominator, then the bigger the fraction must be. This results in pupils then wrongly ordering unit fractions. For example to think that $\frac{1}{3}$ is bigger than $\frac{1}{2}$
2	The magnitude of a fraction depends solely on the denominator and you can ignore the numerator. For example: to think that $\frac{1}{3}$ is bigger than $\frac{4}{5}$.
3	$\frac{3}{4}$ is always more than $\frac{1}{2}$. There is no reference to the quantity.
4	Fractions are added together by adding the numerators together then adding the denominators together. For example to think that $\frac{3}{4} + \frac{1}{2} = \frac{4}{6}$
5	When you multiply fractions the total gets bigger and when you divide they get smaller.
6	Half means just one whole is cut into two pieces. For example – many children will wrongly say that this circle has been cut into thirds. 
7	Fractions of the whole are whole numbers in themselves. For example when an item is cut into half you get two parts (which implies you get more, when in fact it's just 2 halves of the whole, which is less)
8	Fraction symbols incorrectly identified. For example to read $\frac{1}{3}$ as three quarters or even to write three quarters as $3\ {}^{1}/_{4}$ or simply not being able to read fraction symbols at all.

Appendix 50 – Task Design Parameters

Collins (1996) had sought to define tasks in terms of four parameters with exemplifications for each, as can be seen in the table below. The real problem here is these four parameters can be applied to a range of learning episodes and are not specific to tasks.

Goals		
	Memorisation	Thoughtfulness
	Whole Task	Skills
	Breadth of Knowledge	Depth of Knowledge
	Diverse expertise	Uniform expertise
	Access	Understanding
	Cognitive	Physical
Learning Style		
	Interactive	Passive
	Natural	Efficient
	Fun	Serious
	Incidental	Direct
	Learner Controlled	Teacher/Computer controlled
Sequence		
	Grounded	Abstract
	Structured	Exploratory
	Systematic	Diverse
	Simple	Complex
Methods		
	Modelling	
	Scaffolding	
	Coaching	
	Articulation	
	Reflection	

Taken from Collins (1996) Design Issues in Learning Environments

Table 3.3.1 - Brief teacher Biographies
(for members of the mathematics department in the study school).

Teacher	Experience	Commentary
A - Female	2 years	All teaching experience in the school – mathematics degree, trained through the partnership PGCE route
B - Male	4 years	All teaching experience in the school – mathematics degree, trained through the partnership PGCE route
C- Male	6 years	Second school after 1 year in another school. Needed a mathematics conversion course due to a degree in theatre management, trained through the partnership PGCE route
D - Female	Newly qualified	First school (also placement school whilst training). Needed a mathematics conversion course due to a degree in Art and Design, trained through the partnership PGCE route
E - Female	7 years	First school (also placement school whilst training). Needed a mathematics conversion course due to a degree in Education Studies and Psychology, trained through the partnership PGCE route
F - Male	6 years	First school (also placement school whilst training). Needed a mathematics conversion course due to a degree in Engineering, trained through the partnership PGCE route
G - Female	1 year	First school (also placement school whilst training). Needed a mathematics conversion course due to a degree in Psychology and Criminal Justice, trained through the partnership. PGCE route
H - Female	3 years	Second school after 1 year in another school. Needed a mathematics conversion course due to a degree in Accounting and Finance, trained through the partnership PGCE route
I - Female	7 years	First school (also placement school whilst training). Needed a mathematics conversion course due to a degree in Psychology a, trained through the partnership GTP – a school based route
J - Female	10+ years	Not the first school. Degree in Civil Engineering did not train through the partnership (PGCE Route)
K - Female	8 years	First school Degree in Finance – did not train through the partnership PGCE Route)

Table 4.2.2 – Biographies – teaching experience
(of the participating teachers in the study school).

Teacher	Teaching Experience	Commentary
C - Male	6 years	Mature entrant to the profession with good GCSEs but no A levels and no academic mathematical background. He had previously worked in technical theatre for thirteen years, and for five as an arts technician supporting teaching in the arts in a school.
D - Female	Newly qualified	Mature entrant to the profession with 2 year's experience of working as a Sustainable Education Coordinator after leaving university. She has good GCSEs, 2 'A' levels one of which was Mathematics at grade D and the other being Art.
F – Male	6 years	Mature entrant to the profession with 15 year's experience of working in various civil engineering posts. He has good GCSEs, no 'A' levels but a third of modules of the Engineering degree was deemed to be mathematical.
G - Female	1 year	Entered immediately after completing her degree with little or no experience of the classroom. She has good GCSEs, 3 'A' levels and 1 AS level which was Mathematics at grade C.
H - Female	3 years	Mature entrant to the profession after completing her degree with little or no experience of the classroom. She has good GCSEs, 4 'A' levels one of which was Mathematics at grade C. Prior to applying for teacher training she had been working as a sport leader in schools.

Tables 4.3.2 - Overview of tables 4.3.2a to 4.3.2e

The next set of three tables consists of the results from the questionnaire frequency responses for all 201 PGCE trainee teachers

Table 4.3.2a	by Gender (n = 201).
Table 4.3.2b	by Mathematics Degree qualification (n = 96).
Table 4.3.2c	by Non Mathematics Degree qualification (n = 105).
Table 4.3.2d	by age 30 years or younger (n = 155).
Table 4.3.2e	by age over 30 years old (n = 46).

Table 4.3.2a - Questionnaire raw responses by Gender (n = 201)

			FEMALE (n = 122);					MALE (n = 79)				
Qu. Number	Category	Question Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	0	12	33	53	24	0	5	28	37	9
2	S	I think learners gain mathematical insight from practising skills.	2	28	35	39	18	1	25	16	32	5
3	T	I think learners should mainly work on their own when practising skills.	15	47	47	13	0	3	36	30	8	2
4	T	I think learners should tackle tasks.	0	8	30	49	35	0	6	22	36	15
5	T	I think good mathematical tasks are difficult to design.	3	37	42	33	7	0	30	17	26	6
6	T	I think good mathematical tasks have unforeseen learning outcomes.	2	24	38	45	13	0	12	23	33	11
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	44	60	11	4	3	13	45	12	8	1
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	21	88	10	3	0	7	55	12	4	1
9	T	I think tasks are coursework in disguise.	50	52	12	7	1	31	35	7	5	1
10	T	I think planning mathematical tasks is time consuming.	2	36	34	41	9	1	16	27	23	12
11	T	I think mathematical tasks take up too much teaching time.	36	56	23	7	0	19	32	19	8	1
12	T	I think mathematical tasks motivate learners.	0	6	28	60	28	3	6	18	42	10
13	T	I think designing mathematical tasks is a complex exercise.	2	33	34	39	14	0	21	23	26	9
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	21	52	39	10	0	14	29	28	7	1
15	T	I think mathematical tasks do not easily fit the current lesson structure	15	54	26	23	4	11	30	20	14	4
16	P	I think learners feel the 3 part lesson supports their learning.	5	20	39	49	9	9	21	20	24	5
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	2	19	32	62	7	1	9	27	36	6
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	15	49	34	22	2	4	26	26	20	3
19	P	I think learners should be informed as to the learning objective when tackling tasks.	0	21	35	41	25	6	8	17	30	18
20	P	I think learners should learn mathematics through discussing their ideas.	0	2	15	54	51	1	6	12	37	23
21	P	I think learner to learner dialogue (about mathematics) is important.	0	1	11	36	74	1	10	0	38	30
22	P	I think learners should compare and share their solutions.	0	1	6	39	76	0	1	6	40	32
23	P	I think learners should mainly work in pairs or groups.	0	11	55	47	9	0	11	34	29	5
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	2	11	55	33	21	0	8	23	34	14
25	T	I think it is easy to differentiate tasks for pupils	9	42	45	23	3	9	36	21	11	2
26	T	I think differentiating tasks for pupils is a good idea.	0	3	8	42	69	0	2	6	28	43

No Score	Highest Frequency

Table 4.3.2b – Responses from trainees with a mathematics degree. (n = 96).

Question Number	Question Category	Question Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	62	1	9	19	5
2	S	I think learners gain mathematical insight from practising skills.	0	14	22	47	13
3	T	I think learners should mainly work on their own when practising skills.	2	30	26	25	13
4	T	I think learners should tackle tasks.	7	24	36	20	9
5	T	I think good mathematical tasks are difficult to design.	0	17	19	39	21
6	T	I think good mathematical tasks have unforeseen learning outcomes.	2	21	39	25	9
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	8	29	26	24	9
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	25	60	7	2	2
9	T	I think tasks are coursework in disguise.	25	58	10	2	1
10	T	I think planning mathematical tasks is time consuming.	26	34	19	13	4
11	T	I think mathematical tasks take up too much teaching time.	9	34	28	19	6
12	T	I think mathematical tasks motivate learners.	21	33	15	23	4
13	T	I think designing mathematical tasks is a complex exercise.	0	9	21	46	20
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	7	30	33	20	6
15	T	I think mathematical tasks do not easily fit the current lesson structure	22	39	25	9	1
16	P	I think learners feel the 3 part lesson supports their learning.	12	39	18	24	3
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	4	12	36	38	6
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2	17	32	41	4
19	P	I think learners should be informed as to the learning objective when tackling tasks.	10	26	20	27	13
20	P	I think learners should learn mathematics through discussing their ideas.	0	15	20	37	24
21	P	I think learner to learner dialogue (about mathematics) is important.	1	1	14	44	36
22	P	I think learners should compare and share their solutions.	1	1	11	34	49
23	P	I think learners should mainly work in pairs or groups.	0	8	14	36	38
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	0	11	39	38	8
25	T	I think it is easy to differentiate tasks for pupils	3	23	35	24	11
26	T	I think differentiating tasks for pupils is a good idea.	8	22	18	27	21

Table 4.3.2c – Responses from trainees with a non mathematics degree (n = 105).

Question Number	Question Category	Question Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	0	12	36	41	16
2	S	I think learners gain mathematical insight from practising skills.	2	29	29	35	10
3	T	I think learners should mainly work on their own when practising skills.	10	45	41	8	1
4	T	I think learners should tackle tasks.	0	8	28	47	22
5	T	I think good mathematical tasks are difficult to design.	1	38	27	32	7
6	T	I think good mathematical tasks have unforeseen learning outcomes.	2	19	30	43	11
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	27	53	12	11	2
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	13	75	12	5	0
9	T	I think tasks are coursework in disguise.	43	43	12	6	1
10	T	I think planning mathematical tasks is time consuming.	2	29	23	40	11
11	T	I think mathematical tasks take up too much teaching time.	26	40	31	7	1
12	T	I think mathematical tasks motivate learners.	2	9	27	49	18
13	T	I think designing mathematical tasks is a complex exercise.	1	30	25	36	13
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	13	43	38	10	1
15	T	I think mathematical tasks do not easily fit the current lesson structure	13	43	38	10	1
16	P	I think learners feel the 3 part lesson supports their learning.	7	25	30	34	9
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	2	16	29	52	6
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	10	42	27	22	4
19	P	I think learners should be informed as to the learning objective when tackling tasks.	4	15	33	33	20
20	P	I think learners should learn mathematics through discussing their ideas.	0	4	13	47	41
21	P	I think learner to learner dialogue (about mathematics) is important.	0	0	10	38	57
22	P	I think learners should compare and share their solutions.	0	0	6	43	56
23	P	I think learners should mainly work in pairs or groups.	0	9	49	40	7
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	2	7	42	34	20
25	T	I think it is easy to differentiate tasks for pupils	7	42	39	15	2
26	T	I think differentiating tasks for pupils is a good idea.	0	1	9	35	60

Table 4.3.2d – Responses from trainees aged 30 years or younger (n = 155).

Question Number	Category	Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	0	11	47	71	26
2	S	I think learners gain mathematical insight from practising skills.	1	39	35	62	18
3	T	I think learners should mainly work on their own when practising skills.	13	60	64	17	1
4	T	I think learners should tackle tasks.	0	8	41	66	40
5	T	I think good mathematical tasks are difficult to design.	0	11	47	71	26
6	T	I think good mathematical tasks have unforeseen learning outcomes.	1	39	35	62	18
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	13	60	64	17	1
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	0	8	41	66	40
9	T	I think tasks are coursework in disguise.	64	68	15	6	2
10	T	I think planning mathematical tasks is time consuming.	3	43	53	40	16
11	T	I think mathematical tasks take up too much teaching time.	41	70	32	11	1
12	T	I think mathematical tasks motivate learners.	3	9	36	77	30
13	T	I think designing mathematical tasks is a complex exercise.	2	46	43	47	17
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	28	58	57	11	1
15	T	I think mathematical tasks do not easily fit the current lesson structure	19	59	38	32	7
16	P	I think learners feel the 3 part lesson supports their learning.	12	30	46	55	12
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	1	21	50	74	9
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	12	54	50	35	4
19	P	I think learners should be informed as to the learning objective when tackling tasks.	5	22	39	57	32
20	P	I think learners should learn mathematics through discussing their ideas.	0	5	22	70	58
21	P	I think learner to learner dialogue (about mathematics) is important.	0	1	17	58	79
22	P	I think learners should compare and share their solutions.	0	1	10	61	83
23	P	I think learners should mainly work in pairs or groups.	0	19	68	56	12
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	1	13	63	52	26
25	T	I think it is easy to differentiate tasks for pupils	14	64	46	26	5
26	T	I think differentiating tasks for pupils is a good idea.	0	3	10	53	89

Table 4.3.2e – Responses for all trainees aged over 30 years old (n = 46).

Question Number	Category	Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	0	6	14	19	7
2	S	I think learners gain mathematical insight from practising skills.	2	14	16	9	5
3	T	I think learners should mainly work on their own when practising skills.	5	23	13	4	1
4	T	I think learners should tackle tasks.	0	6	11	19	10
5	T	I think good mathematical tasks are difficult to design.	1	15	15	12	3
6	T	I think good mathematical tasks have unforeseen learning outcomes.	1	10	14	17	4
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	8	32	4	2	0
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	6	31	8	1	0
9	T	I think tasks are coursework in disguise.	17	19	4	6	0
10	T	I think planning mathematical tasks is time consuming.	0	9	8	24	5
11	T	I think mathematical tasks take up too much teaching time.	14	18	10	4	0
12	T	I think mathematical tasks motivate learners.	0	3	10	25	8
13	T	I think designing mathematical tasks is a complex exercise.	0	8	14	18	6
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	7	23	10	6	0
15	T	I think mathematical tasks do not easily fit the current lesson structure	7	25	8	5	1
16	P	I think learners feel the 3 part lesson supports their learning.	2	11	13	18	2
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	2	7	9	24	4
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	7	21	10	7	1
19	P	I think learners should be informed as to the learning objective when tackling tasks.	1	7	13	14	11
20	P	I think learners should learn mathematics through discussing their ideas.	1	3	5	21	16
21	P	I think learner to learner dialogue (about mathematics) is important.	1		4	16	25
22	P	I think learners should compare and share their solutions.	1	2	18	0	25
23	P	I think learners should mainly work in pairs or groups.	0	3	21	20	2
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	1	6	15	15	9
25	T	I think it is easy to differentiate tasks for pupils	4	14	20	8	0
26	T	I think differentiating tasks for pupils is a good idea.	0	2	4	27	23

This next set of five tables consists of details from the questionnaire results for

Table 4.3.4a	Trainee respondents from fieldwork school whilst training (n=6)
Table 4.3.4b	Trainee respondents from fieldwork school when qualified and in post (n=6)
Table 4.3.4c	The departmental subject leaders (n=3)
Table4.3.4d	All department members (n=9)
Table 4.3.4e	All results combined

Highlighting (in yellow) indicates where question modal values are at the two extremes of the Likert Scale (either : Almost Never or Almost always)

Table 4.3.4a - Trainee respondents from Study school whilst training (n=6).

Qu	Cat	Question Text	Modal Ans.	Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	4	1.0328
2	S	I think learners gain mathematical insight from practising skills.	3	1.0488
3	S	I think learners should mainly work on their own when practising skills.	2	1.0328
4	T	I think learners should tackle tasks.	4	1.1690
5	T	I think good mathematical tasks are difficult to design.	2	0.8165
6	T	I think good mathematical tasks have unforeseen learning outcomes.	3	0.8367
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	2	1.0954
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	2	0.6325
9	T	I think tasks are coursework in disguise.	2	0.5164
10	T	I think planning mathematical tasks is time consuming.	2	0.8367
11	T	I think mathematical tasks take up too much teaching time.	2	0.5164
12	T	I think mathematical tasks motivate learners.	4	0.6325
13	T	I think designing mathematical tasks is a complex exercise.	3	0.7528
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	2	0.6325
15	T	I think mathematical tasks do not easily fit the current lesson structure	2	0.8367
16	P	I think learners feel the 3 part lesson supports their learning.	2	1.1690
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	3	0.5477
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2	0.8944
19	P	I think learners should be informed as to the learning objective when tackling tasks.	4	1.1690
20	P	I think learners should learn mathematics through discussing their ideas.	5	0.8165
21	P	I think learner to learner dialogue (about mathematics) is important.	4	0.5477
22	P	I think learners should compare and share their solutions.	4	0.5477
23	P	I think learners should mainly work in pairs or groups.	4	0.6325
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	3	1.0328
25	T	I think it is easy to differentiate tasks for pupils	3	0.6325
26	T	I think differentiating tasks for pupils is a good idea.	4	0.5477

Table 4.3.4b - Trainee response from Study School
(when qualified and in post ie during the research; n=6).

Qu	Cat	Question Text	Modal Ans	Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	4	0.9832
2	S	I think learners gain mathematical insight from practising skills.	3	0.7528
3	S	I think learners should mainly work on their own when practising skills.	3	1.0328
4	T	I think learners should tackle tasks.	4	0.7528
5	T	I think good mathematical tasks are difficult to design.	3	0.7528
6	T	I think good mathematical tasks have unforeseen learning outcomes.	4	0.9832
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	2	0.4082
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	1	0.5477
9	T	I think tasks are coursework in disguise.	1	1.3663
10	T	I think planning mathematical tasks is time consuming.	2	1.1690
11	T	I think mathematical tasks take up too much teaching time.	2	0.7528
12	T	I think mathematical tasks motivate learners.	4	0.4082
13	T	I think designing mathematical tasks is a complex exercise.	3	1.0488
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	1	0.8944
15	T	I think mathematical tasks do not easily fit the current lesson structure	2	1.2649
16	P	I think learners feel the 3 part lesson supports their learning.	4	0.9832
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	4	1.0328
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2	1.0328
19	P	I think learners should be informed as to the learning objective when tackling tasks.	2	1.1690
20	P	I think learners should learn mathematics through discussing their ideas.	4	0.4082
21	P	I think learner to learner dialogue (about mathematics) is important.	4	0.7528
22	P	I think learners should compare and share their solutions.	5	0.8165
23	P	I think learners should mainly work in pairs or groups.	3	1.0328
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	4	1.0328
25	T	I think it is easy to differentiate tasks for pupils	2	1.3292
26	T	I think differentiating tasks for pupils is a good idea.	4	1.4720

Table 4.3.4c - Departmental Subject Leaders in the study school (n=3)

Qu	Cat	Question Text	Modal Ans	Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	2	1.0000
2	S	I think learners gain mathematical insight from practising skills.	2	0.5774
3	S	I think learners should mainly work on their own when practising skills.	3	0.5774
4	T	I think learners should tackle tasks.	4	0.5774
5	T	I think good mathematical tasks are difficult to design.	1	1.5275
6	T	I think good mathematical tasks have unforeseen learning outcomes.	2	1.0000
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	1	0.5774
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	2	0.0000
9	T	I think tasks are coursework in disguise.	2	0.5774
10	T	I think planning mathematical tasks is time consuming.	4	1.1547
11	T	I think mathematical tasks take up too much teaching time.	1	1.5275
12	T	I think mathematical tasks motivate learners.	3	1.1547
13	T	I think designing mathematical tasks is a complex exercise.	1	1.0000
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	1	0.5774
15	T	I think mathematical tasks do not easily fit the current lesson structure	1	1.5275
16	P	I think learners feel the 3 part lesson supports their learning.	3	0.5774
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	4	1.1547
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2	0.0000
19	P	I think learners should be informed as to the learning objective when tackling tasks.	2	1.1547
20	P	I think learners should learn mathematics through discussing their ideas.	3	1.1547
21	P	I think learner to learner dialogue (about mathematics) is important.	4	0.5774
22	P	I think learners should compare and share their solutions.	4	0.5774
23	P	I think learners should mainly work in pairs or groups.	3	0.5774
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	4	1.1547
25	T	I think it is easy to differentiate tasks for pupils	3	1.1547
26	T	I think differentiating tasks for pupils is a good idea.	2	1.5275

Table 4.3.4d - Department members in the study school (n=9)

Qu	Cat	Question Text	Modal Ans	Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	4	1.0138
2	S	I think learners gain mathematical insight from practising skills.	3	0.7817
3	S	I think learners should mainly work on their own when practising skills.	3	0.8660
4	T	I think learners should tackle tasks.	4	0.7071
5	T	I think good mathematical tasks are difficult to design.	3	1.0541
6	T	I think good mathematical tasks have unforeseen learning outcomes.	4	0.9280
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	2	0.5000
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	2	0.5000
9	T	I think tasks are coursework in disguise.	2	1.1180
10	T	I think planning mathematical tasks is time consuming.	2	1.0929
11	T	I think mathematical tasks take up too much teaching time.	2	1.0000
12	T	I think mathematical tasks motivate learners.	4	0.6667
13	T	I think designing mathematical tasks is a complex exercise.	3	1.2247
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	1	0.8333
15	T	I think mathematical tasks do not easily fit the current lesson structure	2	1.2693
16	P	I think learners feel the 3 part lesson supports their learning.	2	0.8660
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	4	1.0000
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2	0.8333
19	P	I think learners should be informed as to the learning objective when tackling tasks.	2	1.0929
20	P	I think learners should learn mathematics through discussing their ideas.	4	0.7071
21	P	I think learner to learner dialogue (about mathematics) is important.	4	0.6667
22	P	I think learners should compare and share their solutions.	4	0.7071
23	P	I think learners should mainly work in pairs or groups.	3	0.8660
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	4	1.0138
25	T	I think it is easy to differentiate tasks for pupils	2	1.2693
26	T	I think differentiating tasks for pupils is a good idea.	4	1.3944

Table 4.3.4e – Combined results
for the study school departmental members – Modal values

Qu	Cat	Question Text	Trainees	Qualified	Managers	Combined
1	S	I think learners should spend time in every lesson practising mathematics skills.	4	4	2	4
2	S	I think learners gain mathematical insight from practising skills.	3	3	2	3
3	S	I think learners should mainly work on their own when practising skills.	2	3	3	3
4	T	I think learners should tackle tasks.	4	4	4	4
5	T	I think good mathematical tasks are difficult to design.	2	3	1	3
6	T	I think good mathematical tasks have unforeseen learning outcomes.	3	4	2	4
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	2	2	1	2
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	2	1	2	2
9	T	I think tasks are coursework in disguise.	2	1	2	2
10	T	I think planning mathematical tasks is time consuming.	2	2	4	2
11	T	I think mathematical tasks take up too much teaching time.	2	2	1	2
12	T	I think mathematical tasks motivate learners.	4	4	3	4
13	T	I think designing mathematical tasks is a complex exercise.	3	3	1	3
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	2	1	1	1
15	T	I think mathematical tasks do not easily fit the current lesson structure	2	2	1	2
16	P	I think learners feel the 3 part lesson supports their learning.	2	4	3	2
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	3	4	4	4
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2	2	2	2
19	P	I think learners should be informed as to the learning objective when tackling tasks.	4	2	2	2
20	P	I think learners should learn mathematics through discussing their ideas.	5	4	3	4
21	P	I think learner to learner dialogue (about mathematics) is important.	4	4	4	4
22	P	I think learners should compare and share their solutions.	4	5	4	4
23	P	I think learners should mainly work in pairs or groups.	4	3	3	3
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	3	4	4	4
25	T	I think it is easy to differentiate tasks for pupils	3	2	3	2
26	T	I think differentiating tasks for pupils is a good idea.	4	4	2	4

Table 4.4.1 - Modal values – gender

Males (n = 79) and Females (n=122)

Qu	Cat	Text	Males Mode	Males Std. Deviation	Females Mode	Females Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	4	0.7709	4	0.8911
2	S	I think learners gain mathematical insight from practising skills.	4	1.0010	4	1.0438
3	S	I think learners should mainly work on their own when practising skills.	2	0.8213	2	0.8451
4	T	I think learners should tackle tasks.	4	0.8506	4	0.8908
5	T	I think good mathematical tasks are difficult to design.	2	1.0076	3	0.9529
6	T	I think good mathematical tasks have unforeseen learning outcomes.	4	0.9170	4	0.9700
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	2	0.8908	2	0.8903
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	2	0.7229	2	0.5947
9	T	I think tasks are coursework in disguise.	2	0.9162	2	0.8879
10	T	I think planning mathematical tasks is time consuming.	3	1.0150	4	0.9877
11	T	I think mathematical tasks take up too much teaching time.	2	0.9768	2	0.8479
12	T	I think mathematical tasks motivate learners.	4	0.9361	4	0.8071
13	T	I think designing mathematical tasks is a complex exercise.	4	0.9891	4	1.0310
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	2	0.9257	2	0.8536
15	T	I think mathematical tasks do not easily fit the current lesson structure	2	1.0896	2	1.0364
16	P	I think learners feel the 3 part lesson supports their learning.	4	1.1361	4	0.9697
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	4	0.8448	4	0.8813
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2	0.9687	2	0.9790
19	P	I think learners should be informed as to the learning objective when tackling tasks.	4	1.1724	4	1.0034
20	P	I think learners should learn mathematics through discussing their ideas.	4	0.9323	4	0.7363
21	P	I think learner to learner dialogue (about mathematics) is important.	4	0.7624	5	0.6953
22	P	I think learners should compare and share their solutions.	4	0.6671	5	0.6305
23	P	I think learners should mainly work in pairs or groups.	3	0.8012	3	0.7611
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	4	0.8850	3	0.9381
25	T	I think it is easy to differentiate tasks for pupils	2	0.9591	3	0.9321
26	T	I think differentiating tasks for pupils is a good idea.	5	0.7443	5	0.7284

Table 4.4.2 - Modal values by Degree Qualification

Mathematics Degree (n=96) and Non mathematics degree (n=105)

Qu	Cat	Text	Mode (Maths Degree)	Std. Deviation (Maths Degree)	Mode (Non Maths Degree)	Std. Deviation (Non Maths Degree)
1	S	I think learners should spend time in every lesson practising mathematics skills.	1	1.4216	4	0.8856
2	S	I think learners gain mathematical insight from practising skills.	4	0.8988	4	1.0162
3	S	I think learners should mainly work on their own when practising skills.	2	1.0857	2	0.8097
4	T	I think learners should tackle tasks.	3	1.0662	4	0.8626
5	T	I think good mathematical tasks are difficult to design.	4	1.0122	2	0.9887
6	T	I think good mathematical tasks have unforeseen learning outcomes.	3	0.9549	4	0.9668
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	2	1.1281	2	0.9776
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	2	0.7769	2	0.6521
9	T	I think tasks are coursework in disguise.	2	0.7351	1	0.9070
10	T	I think planning mathematical tasks is time consuming.	2	1.1378	4	1.0423
11	T	I think mathematical tasks take up too much teaching time.	2	1.0682	2	0.9271
12	T	I think mathematical tasks motivate learners.	2	1.1958	4	0.9231
13	T	I think designing mathematical tasks is a complex exercise.	4	0.8780	4	1.0443
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	3	1.0285	2	0.8663
15	T	I think mathematical tasks do not easily fit the current lesson structure	2	0.9515	2	1.0833
16	P	I think learners feel the 3 part lesson supports their learning.	2	1.0841	4	1.0804
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	4	0.9212	4	0.8856
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	4	0.8816	2	1.0296
19	P	I think learners should be informed as to the learning objective when tackling tasks.	4	1.2333	3	1.0750
20	P	I think learners should learn mathematics through discussing their ideas.	4	1.0103	4	0.7978
21	P	I think learner to learner dialogue (about mathematics) is important.	4	0.7947	5	0.6648
22	P	I think learners should compare and share their solutions.	5	0.8056	5	0.6060
23	P	I think learners should mainly work in pairs or groups.	5	0.9366	3	0.7449
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	3	0.8062	3	0.9364
25	T	I think it is easy to differentiate tasks for pupils	3	1.0259	2	0.8768
26	T	I think differentiating tasks for pupils is a good idea.	4	1.2773	5	0.6943

Table 4.4.3 - Modal values for PGCE mentors (n = 21)

Qu	Cat	Question Text	Modal Ans	Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	5	0.8729
2	S	I think learners gain mathematical insight from practising skills.	3	0.9562
3	S	I think learners should mainly work on their own when practising skills.	3	0.6583
4	T	I think learners should tackle tasks.	4	1.0757
5	T	I think good mathematical tasks are difficult to design.	3	0.8452
6	T	I think good mathematical tasks have unforeseen learning outcomes.	5	1.2440
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	1	1.0623
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	2	0.3008
9	T	I think tasks are coursework in disguise.	2	0.9437
10	T	I think planning mathematical tasks is time consuming.	2	0.8891
11	T	I think mathematical tasks take up too much teaching time.	2	0.9129
12	T	I think mathematical tasks motivate learners.	4	1.0646
13	T	I think designing mathematical tasks is a complex exercise.	2	0.9636
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	2	1.0142
15	T	I think mathematical tasks do not easily fit the current lesson structure	2	1.0757
16	P	I think learners feel the 3 part lesson supports their learning.	4	1.2440
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	4	1.0235
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	3	1.1360
19	P	I think learners should be informed as to the learning objective when tackling tasks.	4	1.1670
20	P	I think learners should learn mathematics through discussing their ideas.	3	0.9562
21	P	I think learner to learner dialogue (about mathematics) is important.	5	0.9103
22	P	I think learners should compare and share their solutions.	5	0.6796
23	P	I think learners should mainly work in pairs or groups.	3	0.7746
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	4	0.8309
25	T	I think it is easy to differentiate tasks for pupils	4	0.9661
26	T	I think differentiating tasks for pupils is a good idea.	5	0.9207

Table 4.5.1 - Selected quotations from 7 of the 14 lesson plans.

Designed before the research highlighting the use of the words "activity" and "task".

Teacher	Quotations from the lesson plans	Teacher beliefs from the questionnaire pre qualifying
A	**LESSON PLAN 2** (context – working out circumferences of circles). Extension **activity** provided for those who are working well and understanding the **task**.	Tasks should be used all the time
B	**LESSON PLAN 2** - **Activity** 1: Worksheet on ratio to peer assess. Extension **task** – simplifying ratio bingo on power point if plenty time.	Tasks should be used all the time as they are a motivator.
D	**LESSON PLAN 1** – Use clues to deduce age of man – give 5 minutes for the **task**	Tasks should be used all the time they give unforeseen outcomes which motivate pupils.
	LESSON PLAN 2 – Drawing lines of symmetry – check pupils are on **task**	
	LESSON PLAN 4 – Extension – match-up **activity**	
F	**LESSON PLAN 4**– Continuing a sequence – starter **activity**. Classifying triangles – handout additional hints and extension **tasks**	Tasks should be used occasionally
G	**LESSON PLAN 1 – Task** – by the end of the next 15 minutes I would like to know the chances / probabilities of picking each colour	Tasks should be used which give unforeseen outcomes which motivate pupils.

Table 4.5.2 - Quotations from 4 of the 5 post research lesson plans highlighting the use of the words activity, exercise, skill and task

Teacher	Quotations from the lesson plans
C	**Activity – In pairs -** Using a tarsia puzzle to recall knowledge of how to solve one step linear equations **Task** – **Individually** – Question posed by the teacher:- "Two pupils are trying to decide when the following equation is true: 5 - *x* = 4. One pupil thinks it is true when x = 1 the other thinks it is never true." Can you help?
D	**Activity** – **In tables (4s)** – Produce a list of all the properties of quadrilaterals – support materials available **Task** – **in Pairs** – Pupils are given a pack of shapes to sort – they are required to place them in categories and find the shapes that do not 'belong' and describe their properties
G	**Activity – In six teams –** research / recall all the formulas we know for volumes of solids. Produce a formula sheet. **Task** – **in six teams** – A competition design and build a robot from a pack of 3D solids (eg cornflakes boxes) calculating the volumes. Your task is to build a robot with a volume as close to $5000cm^3$ as possible.
H	**Activity – Individually** - Using a prepared worksheet plotting 4 quadrant co-ordinates to form a well-known given logo. **Task** – **Two teams** – A competition – the room is set out as a 3 dimensional grid 3 x 3 x 3 to play 3 dimensional noughts – and crosses using co-ordinates to specify the location of their marker.

Table 4.6a - Pupil Views - where given in 7AC *(spelling corrected)*

Pupil	Q1 . When you work with the fractions tiles how did they help you understand fractions
A1	The made understand how many lower fractions go in the bigger one
A2	They helped me understand what goes into what
B1	They helped me to understand because I was physically able to do it
B2	They helped me with the GCSE question [Worksheet 3]
C1	They helped because it had a visual and I could see what when in the GCSE question [Worksheet 3]
C2	They helped me with the hard questions
D1	They helped me with the hard question which made me revise how to look at frae1ctions
D2	They showed me how many different fractiog1ns go into a whole which helped when doing the questions
E1	They helped me understand how the denominator is x2 every time we halved the tile
E2	They helped me because it had a visual effect and you could see how many mad a whole.
F1	It helped me because I could see the fractions so it was easier
F2	Yes because you have the shape made ready
G1	Because I could actually see the fractions
G2	Yes, because I could actually see the fractions
H1	Yes, It helped me because you can see the fractions. You can put them together too.
H2	Practical based get us active and doing it
G1	It made it easier because I could visually see what I needed to work out.
G2	They helped me understand fractions because they were in fractions
H1	They made me understand how many lower fractions go in the bigger fraction
H2	They helped because I could use them and put them on the full piece [whole][of paper to see if I have covered half or something.
I1	They were a physical representation
J1	They helped me understand halves of fractions
J2	They helped me OK, but once I used the tiles once the I go on
K1	The made understand how many lower fractions go in the bigger one

Pupil	Q2 How easy did you find the making up of your own "What if questions"?
A1	Easy
B1	I found it quite easy
B2	Easy
C2	I didn't do it but if I did I would be okay
E1	Easy
F1	It was a little bit easy
G1	Very
H2	Quite easy
G1	I found it easy because I had the example questions to refer to
G2	I found it pretty easy
H1	Yes, easy
H2	I found it very easy
I1	Didn't do it
I2	Didn't do it
J1	I found them quite easy to make up
J2	Very easy

Pupil	Q3 How easy was the last tasks finding the answer to the squash problem?
A1	At the start it was complicated but (Pupil Name) made me get it
A2	I was hard but after sir explained it again
B1	I found it actually quite easy
B2	It wasn't too hard
C2	Quite hard to be fair
E1	Easy
H2	Simple
G1	It was easy once I knew what I was doing
G2	It was a little hard
H1	At the start it was frustrating but now it is easy
H2	I also found it quite easy
I1	Very
I2	It was hard and easy as we was given the harder sheet [sheet 3b]
J1	It was actually quite easy
J2	Easy
K1	At the start it was complicated but Jason made me get it

Pupil	Q4 Most of the lesson involved you working with another pupil. How did this help you understand fractions?
A2	She understood me than me so it was a help
B2	I listened to her ideas
C2	She talked to me about it when I got stuck
E1	We worked together to complete the questions
H2	Yes because I like teamwork
G1	Because we both put our ideas together
G2	Because I helped her understand some questions and she helped me
H1	Because we have our own opinion
I1	It gave an extra mind to work the problem out.
J1	Because when I didn't understand something Courtney did.
J2	Very well because me and Amber are good friends

Pupil	Q5 Which part of the lesson do you need to do more work on?
B2	Last question
C2	The sorting out the tiles
E1	The squash problem
G2	The last task
I1	None
I2	None

Table 4.6b - Pupil Views –from pupils in 7NR *(spelling corrected)*

Pupil	Q1 . When you work with the fractions tiles how did they help you understand fractions
A1	They helped because I could explain better
A2	Because we made it easier
B1	It helped me that I knew more fraction
B2	It helped because I could experiment with them
C1	Because it's easy
C2	They helped me because I know how many is in a half and a quarter ....
D1	We could use them to answer the questions
D2	Being using objects physically
E1	We found the questions a lot easier with the tiles
E2	Because it helped by getting me more confident
F1	They helped me understand fractions more and there easier to use
F2	They helped me very well with counting them up
G1	They helped me understand because I could see them and help me understand it is better this way I think
G2	They helped to show what they stood for
H1	They helped by putting them together to make the right fraction
H2	They helped me to see how many fractions were in a fraction

Pupil	Q2 How easy did you find the making up of your own "What if questions"?
A1	It was so easy to work out
A2	A [???] bit hard
B1	It was a bit difficult
B2	I found it hard because I couldn't work it out
C1	It was a bit hard
C2	It was hard
D1	It was easy when we worked together
D2	Quite easy and challenging
E1	I didn't find it easy
E2	
F1	I didn’t get time to make up my own but If I did find the questions easy
F2	it was harder than having to solve them
G1	I think it was a bit harder then answering one but me an my partner did make one
G2	It was kinda hard due to I didn't think of any idea for it
H1	I didn't have time to make up my own but I did find he questions easy
H2	I thought it was easy

Pupil	Q3 How easy was the last tasks finding the answer to the squash problem?
A1	It was easy because of the tiles
A2	Didn't do it
B1	It was hard bit after it was good
B2	Easy because I go to add up all the numbers
C1	Didn't do it
C2	It was a bit hard
D1	It was easy when it was explained to me
D2	It was awesome really easy thinking ?? Maths
E1	I found it quite easy
E2	It was quite hard but easy when we got help
F1	It was actually really fun, easy, interesting and challenging
F2	It was not easy but it was not hard
G1	It was easier with the tiles but it was a little difficult
G2	Challenging
H1	It was easy working with my partner
H2	I thought it was easy

Pupil	Q4 Most of the lesson involved you working with another pupil. How did this help you understand fractions?
A1	Things I did not [k][now, she helped me with
A2	Because we had someone to help
B1	Because we worked as a team and the tiles
B2	It helped because I’m better in paired groups
C1	Because we had someone to help
C2	I worked with on my own
D1	We could work together
D2	Not really I worked out all the questions
E1	I find it easy with other pupils than working on my own.
E2	We worked as a team
F1	This helped out very well with fractions
F2	It helped a lot. We both used our knowledge to solve it.
G1	How many of the other smaller fractions fit in the bigger one
G2	It helped
H1	This helped because she helped me and I did other things
H2	I worked on my own

Pupil	Q5 Which part of the lesson do you need to do more work on?
A1	The last worksheet
A2	The solving questions
B1	The last question
B2	The ending question (What if)
C1	Solving questions
C2	The squash part
D1	*(No response from the pupil)*
D2	None I think I was good
E1	*(No response from the pupil)*
E2	More on fractions itself
F1	*(No response from the pupil)*
F2	Our own what if questions
G1	The last part with the concentrated squash and my own questions
G2	*(No response from the pupil)*
H1	I was good at everything
H2	Nothing

Table 4.7 - Responses of the participant teachers.

whilst training and then again when qualified working in the study school (n = 6)

	View for trainees whilst on the PGCE Course.					Results for trainees once qualified (in the study school)				
Question Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
I think learners should spend time in every lesson practising mathematics skills.	0	1	1	3	1	0	1	0	4	1
I think learners gain mathematical insight from practising skills.	0	1	2	2	1	0	1	3	2	0
I think learners should mainly work on their own when practising skills.	1	3	1	1	0	1	1	3	1	0
I think learners should tackle tasks.	0	1	1	2	2	0	0	1	3	2
I think good mathematical tasks are difficult to design.	0	3	2	1	0	0	1	3	2	0
I think good mathematical tasks have unforeseen learning outcomes.	0	0	4	1	1	0	2	1	3	0
I think good mathematical tasks can lead learners along unproductive pathways.	2	1	0	1	0	1	5	0	0	0
I think mathematical tasks can lead learners to incorrect mathematical conclusions.	1	4	1		0	3	3	0	0	0
I think tasks are coursework in disguise.	2	4	0	0	0	2	2	2	0	0
I think planning mathematical tasks is time consuming.	0	4	1	1	0	0	2	2	1	1
I think mathematical tasks take up too much teaching time.	2	4	0	0	0	1	3	2	0	0
I think mathematical tasks motivate learners.	0	0	1	4	1	0	0	1	5	0
I think designing mathematical tasks is a complex exercise.	0	2	3	1	0	0	1	2	2	1
I think designing mathematical tasks is not necessary as other resources are readily available.	1	4	1	0	0	2	2	2	0	0
I think mathematical tasks do not easily fit the current lesson structure	0	4	1	1	0	0	3	1	1	1
I think learners feel the 3 part lesson supports their learning.	0	2	2	1	1	0	2	1	3	0
I think learners feel confident and secure if I use one or two teaching approaches.	0	0	3	3	0	0	2	0	4	0
I think learners feel confident and secure if I only use one or two teaching approaches.	0	2	2	2	0	1	3	1	1	0
I think learners should be informed as to the learning objective when tackling tasks.	0	1	1	2	2	0	3	2	1	0
I think learners should learn mathematics through discussing their ideas.	0	0	1	2	3	0	0	0	5	1
I think learner to learner dialogue (about mathematics) is important.	0	0	0	3	3	0	0	1	3	2
I think learners should compare and share their solutions.	0	0	0	3	3	0	0	1	2	3
I think learners should mainly work in pairs or groups.	0	0	1	4	1	0	1	3	1	1
I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	0	1	3	1	1	0	1	1	3	1
I think it is easy to differentiate tasks for pupils	0	1	4	1	0	0	4	0	1	1
I think differentiating tasks for pupils is a good idea.	0	0	0	3	3	0	0	0	3	1

Table 4.8 - Frequency Table comparison of trainees views (n = 6)

	View for trainees whilst on the PGCE Course.					Results for trainees once qualified (in the case school)				
Question Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
I think learners should spend time in every lesson practising mathematics skills.	0	1	1	3	1	0	1	0	4	1
I think learners gain mathematical insight from practising skills.	0	1	2	2	1	0	1	3	2	0
I think learners should mainly work on their own when practising skills.	1	3	1	1	0	1	1	3	1	0
I think learners should tackle tasks.	0	1	1	2	2	0		1	3	2
I think good mathematical tasks are difficult to design.	0	3	2	1	0	0	1	3	2	0
I think good mathematical tasks have unforeseen learning outcomes.	0	0	4	1	1	0	2	1	3	0
I think good mathematical tasks can lead learners along unproductive pathways.	2	1	0	1	0	1	5	0	0	0
I think mathematical tasks can lead learners to incorrect mathematical conclusions.	1	4	1		0	3	3	0	0	0
I think tasks are coursework in disguise.	2	4	0	0	0	2	2	2	0	0
I think planning mathematical tasks is time consuming.	0	4	1	1	0	0	2	2	1	1
I think mathematical tasks take up too much teaching time.	2	4	0	0	0	1	3	2	0	0
I think mathematical tasks motivate learners.	0	0	1	4	1	0		1	5	0
I think designing mathematical tasks is a complex exercise.	0	2	3	1	0	0	1	2	2	1
I think designing mathematical tasks is not necessary as other resources are readily available.	1	4	1	0	0	2	2	2	0	0
I think mathematical tasks do not easily fit the current lesson structure	0	4	1	1	0	0	3	1	1	1
I think learners feel the 3 part lesson supports their learning.	0	2	2	1	1	0	2	1	3	0
I think learners feel confident and secure if I use one or two teaching approaches.	0	0	3	3	0	0	2	0	4	0
I think learners feel confident and secure if I only use one or two teaching approaches.	0	2	2	2	0	1	3	1	1	0
I think learners should be informed as to the learning objective when tackling tasks.	0	1	1	2	2	0	3	2	1	0
I think learners should learn mathematics through discussing their ideas.	0	0	1	2	3	0	0	0	5	1
I think learner to learner dialogue (about mathematics) is important.	0	0	0	3	3	0	0	1	3	2
I think learners should compare and share their solutions.	0	0	0	3	3	0	0	1	2	3
I think learners should mainly work in pairs or groups.	0	0	1	4	1	0	1	3	1	1
I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	0	1	3	1	1	0	1	1	3	1
I think it is easy to differentiate tasks for pupils	0	1	4	1	0	0	4	0	1	1
I think differentiating tasks for pupils is a good idea.	0	0	0	3	3	0	0	0	3	1

Table 4.9 - Frequency Table All PGCE mentors Respondents (n = 21)

Question Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
I think learners should spend time in every lesson practising mathematics skills.	0	1	3	8	9
I think learners gain mathematical insight from practising skills.	0	2	7	7	5
I think learners should mainly work on their own when practising skills.	0	1	13	6	1
I think learners should tackle tasks.	0	6	3	9	3
I think good mathematical tasks are difficult to design.	2	5	11	3	0
I think good mathematical tasks have unforeseen learning outcomes.	0	6	3	5	7
I think good mathematical tasks can lead learners along unproductive pathways.	9	9	1	1	1
I think mathematical tasks can lead learners to incorrect mathematical conclusions.	2	19	0	0	0
I think tasks are coursework in disguise.	6	9	4	2	0
I think planning mathematical tasks is time consuming.	0	9	5	7	0
I think mathematical tasks take up too much teaching time.	1	10	5	5	0
I think mathematical tasks motivate learners.	1	4	5	9	2
I think designing mathematical tasks is a complex exercise.	0	10	5	5	1
I think designing mathematical tasks is not necessary as other resources are readily available.	1	8	9	5	1
I think mathematical tasks do not easily fit the current lesson structure	4	8	6	2	1
I think learners feel the 3 part lesson supports their learning.	2	4	2	10	3
I think learners feel confident and secure if I use one or two teaching approaches.	1	2	4	11	3
I think learners feel confident and secure if I only use one or two teaching approaches.	3	4	7	6	1
I think learners should be informed as to the learning objective when tackling tasks.	1	2	4	7	7
I think learners should learn mathematics through discussing their ideas.	0	2	7	7	5
I think learner to learner dialogue (about mathematics) is important.	0	1	4	7	9
I think learners should compare and share their solutions.	0	0	2	6	13
I think learners should mainly work in pairs or groups.	0	5	12	3	1
I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	0	1	7	9	4
I think it is easy to differentiate tasks for pupils	0	5	6	8	2
I think differentiating tasks for pupils is a good idea.	0	2	0	7	12

Tables 4.9a to 4.9v - Raw results for each of the identified groupings

Table	Grouping
4.9a	All trainee respondents; (n = 201)
4.9b	All trainee respondents aged 30 years or younger; (n = 155)
4.9c	All trainee respondents aged over 30 years; (n = 46)
4.9d	All male trainee respondents (n = 79)
4.9e	All female trainee respondents (n = 122)
4.9f	All trainees respondents with a mathematics degree (n =96)
4.9g	All trainee respondents without a mathematics degree; (n = 105)
4.9h	The research school trainee respondents; (n = 6)
4.9i	The research school qualified respondents; (n = 6)
4.9j	The research school subject leaders; (n = 3)
4.9k	All PGCE subject mentors; (n = 21)

Highlighting indicates no score

Descriptive statistics (mode, mean and standard Deviation) for each of the identified groupings

Table	Grouping
4.9l	All trainee respondents; (n = 201)
4.9m	All trainee respondents aged 30 years or younger; (n = 155)
4.9n	All trainee respondents aged over 30 years; (n = 46)
4.9o	All male trainee respondents (n = 79)
4.9p	All female trainee respondents (n = 122)
4.9q	All trainees respondents with a mathematics degree (n =96)
4.9r	All trainee respondents without a mathematics degree; (n = 105)
4.9s	The research school trainee respondents; (n = 6)
4.9t	The research school qualified respondents; (n = 6)
4.9u	The research school subject leaders; (n = 3)
4.9v	All PGCE subject mentors; (n = 21)

Highlighting indicating where modes or mean are the extremes on the Likert Scale (Almost Never or Almost Always).

Table 4.9a - Frequencies Table
All trainee respondents (n = 201)

Qu. No.	Category	Question Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	0	17	61	90	33
2	S	I think learners gain mathematical insight from practising skills.	3	53	51	71	23
3	S	I think learners should mainly work on their own when practising skills.	18	83	77	21	2
4	T	I think learners should tackle tasks.	0	14	52	85	50
5	T	I think good mathematical tasks are difficult to design.	3	67	59	59	13
6	T	I think good mathematical tasks have unforeseen learning outcomes.	2	36	61	78	24
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	57	105	23	12	4
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	28	143	22	7	1
9	T	I think tasks are coursework in disguise.	81	87	19	12	2
10	T	I think planning mathematical tasks is time consuming.	3	52	61	64	21
11	T	I think mathematical tasks take up too much teaching time.	55	88	42	15	1
12	T	I think mathematical tasks motivate learners.	3	12	46	102	38
13	T	I think designing mathematical tasks is a complex exercise.	2	54	57	65	23
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	35	81	67	17	1
15	T	I think mathematical tasks do not easily fit the current lesson structure	26	84	46	37	8
16	P	I think learners feel the 3 part lesson supports their learning.	14	41	59	73	14
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	3	28	59	98	13
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	19	75	60	42	5
19	P	I think learners should be informed as to the learning objective when tackling tasks.	6	29	52	71	43
20	P	I think learners should learn mathematics through discussing their ideas.	1	8	27	91	74
21	P	I think learner to learner dialogue (about mathematics) is important.	1	1	21	74	104
22	P	I think learners should compare and share their solutions.	0	2	12	79	108
23	P	I think learners should mainly work in pairs or groups.	0	22	89	76	14
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	2	19	78	67	35
25	T	I think it is easy to differentiate tasks for pupils	18	78	66	34	5
26	T	I think differentiating tasks for pupils is a good idea.	0	5	14	70	112

Table 4.9b - Frequencies Table Age 1

All respondents aged 30 years or younger (n = 155)

Question Number	Category	Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	0	11	47	71	26
2	S	I think learners gain mathematical insight from practising skills.	1	39	35	62	18
3	T	I think learners should mainly work on their own when practising skills.	13	60	64	17	1
4	T	I think learners should tackle tasks.	0	8	41	66	40
5	T	I think good mathematical tasks are difficult to design.	0	11	47	71	26
6	T	I think good mathematical tasks have unforeseen learning outcomes.	1	39	35	62	18
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	13	60	64	17	1
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	0	8	41	66	40
9	T	I think tasks are coursework in disguise.	64	68	15	6	2
10	T	I think planning mathematical tasks is time consuming.	3	43	53	40	16
11	T	I think mathematical tasks take up too much teaching time.	41	70	32	11	1
12	T	I think mathematical tasks motivate learners.	3	9	36	77	30
13	T	I think designing mathematical tasks is a complex exercise.	2	46	43	47	17
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	28	58	57	11	1
15	T	I think mathematical tasks do not easily fit the current lesson structure	19	59	38	32	7
16	P	I think learners feel the 3 part lesson supports their learning.	12	30	46	55	12
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	1	21	50	74	9
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	12	54	50	35	4
19	P	I think learners should be informed as to the learning objective when tackling tasks.	5	22	39	57	32
20	P	I think learners should learn mathematics through discussing their ideas.	0	5	22	70	58
21	P	I think learner to learner dialogue (about mathematics) is important.	0	1	17	58	79
22	P	I think learners should compare and share their solutions.	0	1	10	61	83
23	P	I think learners should mainly work in pairs or groups.	0	19	68	56	12
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	1	13	63	52	26
25	T	I think it is easy to differentiate tasks for pupils	14	64	46	26	5
26	T	I think differentiating tasks for pupils is a good idea.	0	3	10	53	89

Table 4.9c - Frequency Table Age 2
All trainee respondents age over 30 years old (n = 46)

Question Number	Category	Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	0	6	14	19	7
2	S	I think learners gain mathematical insight from practising skills.	2	14	16	9	5
3	T	I think learners should mainly work on their own when practising skills.	5	23	13	4	1
4	T	I think learners should tackle tasks.	0	6	11	19	10
5	T	I think good mathematical tasks are difficult to design.	1	15	15	12	3
6	T	I think good mathematical tasks have unforeseen learning outcomes.	1	10	14	17	4
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	8	32	4	2	0
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	6	31	8	1	0
9	T	I think tasks are coursework in disguise.	17	19	4	6	0
10	T	I think planning mathematical tasks is time consuming.	0	9	8	24	5
11	T	I think mathematical tasks take up too much teaching time.	14	18	10	4	0
12	T	I think mathematical tasks motivate learners.	0	3	10	25	8
13	T	I think designing mathematical tasks is a complex exercise.	0	8	14	18	6
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	7	23	10	6	0
15	T	I think mathematical tasks do not easily fit the current lesson structure	7	25	8	5	1
16	P	I think learners feel the 3 part lesson supports their learning.	2	11	13	18	2
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	2	7	9	24	4
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	7	21	10	7	1
19	P	I think learners should be informed as to the learning objective when tackling tasks.	1	7	13	14	11
20	P	I think learners should learn mathematics through discussing their ideas.	1	3	5	21	16
21	P	I think learner to learner dialogue (about mathematics) is important.	1		4	16	25
22	P	I think learners should compare and share their solutions.	1	2	18	0	25
23	P	I think learners should mainly work in pairs or groups.	0	3	21	20	2
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	1	6	15	15	9
25	T	I think it is easy to differentiate tasks for pupils	4	14	20	8	0
26	T	I think differentiating tasks for pupils is a good idea.	0	2	4	27	23

Table 4.9d - Frequency Table - Males

All male trainee Respondents (n = 79)

Question Number	Category	Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	0	5	28	37	9
2	S	I think learners gain mathematical insight from practising skills.	1	25	16	32	5
3	T	I think learners should mainly work on their own when practising skills.	3	36	30	8	2
4	T	I think learners should tackle tasks.	0	6	22	36	15
5	T	I think good mathematical tasks are difficult to design.	0	30	17	26	6
6	T	I think good mathematical tasks have unforeseen learning outcomes.	0	12	23	33	11
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	13	45	12	8	1
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	7	55	12	4	1
9	T	I think tasks are coursework in disguise.	31	35	7	5	1
10	T	I think planning mathematical tasks is time consuming.	1	16	27	23	12
11	T	I think mathematical tasks take up too much teaching time.	19	32	19	8	1
12	T	I think mathematical tasks motivate learners.	3	6	18	42	10
13	T	I think designing mathematical tasks is a complex exercise.	0	21	23	26	9
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	14	29	28	7	1
15	T	I think mathematical tasks do not easily fit the current lesson structure	11	30	20	14	4
16	P	I think learners feel the 3 part lesson supports their learning.	9	21	20	24	5
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	1	9	27	36	6
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	4	26	26	20	3
19	P	I think learners should be informed as to the learning objective when tackling tasks.	6	8	17	30	18
20	P	I think learners should learn mathematics through discussing their ideas.	1	6	12	37	23
21	P	I think learner to learner dialogue (about mathematics) is important.	1	10	0	38	30
22	P	I think learners should compare and share their solutions.	0	1	6	40	32
23	P	I think learners should mainly work in pairs or groups.	0	11	34	29	5
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	0	8	23	34	14
25	T	I think it is easy to differentiate tasks for pupils	9	36	21	11	2
26	T	I think differentiating tasks for pupils is a good idea.	0	2	6	28	43

Table 4.9e - Frequencies Table –Females
All female trainee respondents (n = 122)

Question Number	Category	Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	0	12	33	53	24
2	S	I think learners gain mathematical insight from practising skills.	2	28	35	39	18
3	T	I think learners should mainly work on their own when practising skills.	15	47	47	13	0
4	T	I think learners should tackle tasks.	0	8	30	49	35
5	T	I think good mathematical tasks are difficult to design.	3	37	42	33	7
6	T	I think good mathematical tasks have unforeseen learning outcomes.	2	24	38	45	13
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	44	60	11	4	3
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	21	88	10	3	0
9	T	I think tasks are coursework in disguise.	50	52	12	7	1
10	T	I think planning mathematical tasks is time consuming.	2	36	34	41	9
11	T	I think mathematical tasks take up too much teaching time.	36	56	23	7	0
12	T	I think mathematical tasks motivate learners.	0	6	28	60	28
13	T	I think designing mathematical tasks is a complex exercise.	2	33	34	39	14
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	21	52	39	10	0
15	T	I think mathematical tasks do not easily fit the current lesson structure	15	54	26	23	4
16	P	I think learners feel the 3 part lesson supports their learning.	5	20	39	49	9
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	2	19	32	62	7
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	15	49	34	22	2
19	P	I think learners should be informed as to the learning objective when tackling tasks.	0	21	35	41	25
20	P	I think learners should learn mathematics through discussing their ideas.	0	2	15	54	51
21	P	I think learner to learner dialogue (about mathematics) is important.	0	1	11	36	74
22	P	I think learners should compare and share their solutions.	0	1	6	39	76
23	P	I think learners should mainly work in pairs or groups.	0	11	55	47	9
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	2	11	55	33	21
25	T	I think it is easy to differentiate tasks for pupils	9	42	45	23	3
26	T	I think differentiating tasks for pupils is a good idea.	0	3	8	42	69

Table 4.9f – Frequency Table – Degree 1

All trainees Respondents with a Mathematics Degree (n = 96)

Question Number	Category	Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	62	1	9	19	5
2	S	I think learners gain mathematical insight from practising skills.	0	14	22	47	13
3	T	I think learners should mainly work on their own when practising skills.	2	30	26	25	13
4	T	I think learners should tackle tasks.	7	24	36	20	9
5	T	I think good mathematical tasks are difficult to design.	0	17	19	39	21
6	T	I think good mathematical tasks have unforeseen learning outcomes.	2	21	39	25	9
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	8	29	26	24	9
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	25	60	7	2	2
9	T	I think tasks are coursework in disguise.	25	58	10	2	1
10	T	I think planning mathematical tasks is time consuming.	26	34	19	13	4
11	T	I think mathematical tasks take up too much teaching time.	9	34	28	19	6
12	T	I think mathematical tasks motivate learners.	21	33	15	23	4
13	T	I think designing mathematical tasks is a complex exercise.	0	9	21	46	20
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	7	30	33	20	6
15	T	I think mathematical tasks do not easily fit the current lesson structure	22	39	25	9	1
16	P	I think learners feel the 3 part lesson supports their learning.	12	39	18	24	3
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	4	12	36	38	6
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2	17	32	41	4
19	P	I think learners should be informed as to the learning objective when tackling tasks.	10	26	20	27	13
20	P	I think learners should learn mathematics through discussing their ideas.	0	15	20	37	24
21	P	I think learner to learner dialogue (about mathematics) is important.	1	1	14	44	36
22	P	I think learners should compare and share their solutions.	1	1	11	34	49
23	P	I think learners should mainly work in pairs or groups.	0	8	14	36	38
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	0	11	39	38	8
25	T	I think it is easy to differentiate tasks for pupils	3	23	35	24	11
26	T	I think differentiating tasks for pupils is a good idea.	8	22	18	27	21

Table 4.9g – Frequency Table - Degree 2

All trainees Respondents without a Mathematics Degree (n = 105)

Question Number	Category	Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	0	12	36	41	16
2	S	I think learners gain mathematical insight from practising skills.	2	29	29	35	10
3	T	I think learners should mainly work on their own when practising skills.	10	45	41	8	1
4	T	I think learners should tackle tasks.	0	8	28	47	22
5	T	I think good mathematical tasks are difficult to design.	1	38	27	32	7
6	T	I think good mathematical tasks have unforeseen learning outcomes.	2	19	30	43	11
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	27	53	12	11	2
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	13	75	12	5	0
9	T	I think tasks are coursework in disguise.	43	43	12	6	1
10	T	I think planning mathematical tasks is time consuming.	2	29	23	40	11
11	T	I think mathematical tasks take up too much teaching time.	26	40	31	7	1
12	T	I think mathematical tasks motivate learners.	2	9	27	49	18
13	T	I think designing mathematical tasks is a complex exercise.	1	30	25	36	13
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	13	43	38	10	1
15	T	I think mathematical tasks do not easily fit the current lesson structure	13	43	38	10	1
16	P	I think learners feel the 3 part lesson supports their learning.	7	25	30	34	9
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	2	16	29	52	6
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	10	42	27	22	4
19	P	I think learners should be informed as to the learning objective when tackling tasks.	4	15	33	33	20
20	P	I think learners should learn mathematics through discussing their ideas.	0	4	13	47	41
21	P	I think learner to learner dialogue (about mathematics) is important.	0	0	10	38	57
22	P	I think learners should compare and share their solutions.	0	0	6	43	56
23	P	I think learners should mainly work in pairs or groups.	0	9	49	40	7
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	2	7	42	34	20
25	T	I think it is easy to differentiate tasks for pupils	7	42	39	15	2
26	T	I think differentiating tasks for pupils is a good idea.	0	1	9	35	60

Table 4.9h – Frequency Table – Participants 1

Trainees at the study school when on PGCE course (n = 6)

Question Number	Category	Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	0	1	1	3	1
2	S	I think learners gain mathematical insight from practising skills.	0	1	2	2	1
3	T	I think learners should mainly work on their own when practising skills.	1	3	1	1	0
4	T	I think learners should tackle tasks.	0	1	1	2	2
5	T	I think good mathematical tasks are difficult to design.	0	3	2	1	0
6	T	I think good mathematical tasks have unforeseen learning outcomes.	0	0	4	1	1
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	2	1	0	1	0
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	1	4	1		0
9	T	I think tasks are coursework in disguise.	2	4	0	0	0
10	T	I think planning mathematical tasks is time consuming.	0	4	1	1	0
11	T	I think mathematical tasks take up too much teaching time.	2	4	0	0	0
12	T	I think mathematical tasks motivate learners.	0	0	1	4	1
13	T	I think designing mathematical tasks is a complex exercise.	0	2	3	1	0
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	1	4	1	0	0
15	T	I think mathematical tasks do not easily fit the current lesson structure	0	4	1	1	0
16	P	I think learners feel the 3 part lesson supports their learning.	0	2	2	1	1
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	0	0	3	3	0
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	0	2	2	2	0
19	P	I think learners should be informed as to the learning objective when tackling tasks.	0	1	1	2	2
20	P	I think learners should learn mathematics through discussing their ideas.	0	0	1	2	3
21	P	I think learner to learner dialogue (about mathematics) is important.	0	0	0	3	3
22	P	I think learners should compare and share their solutions.	0	0	0	3	3
23	P	I think learners should mainly work in pairs or groups.	0	0	1	4	1
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	0	1	3	1	1
25	T	I think it is easy to differentiate tasks for pupils	0	1	4	1	0
26	T	I think differentiating tasks for pupils is a good idea.	0	0	0	3	3

Table 4.9i – Frequency Table –Participants 2

Qualified trainees at the study school when on PGCE course (n = 6)

Question Number	Category	Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	0	1	0	4	1
2	S	I think learners gain mathematical insight from practising skills.	0	1	3	2	0
3	T	I think learners should mainly work on their own when practising skills.	1	1	3	1	0
4	T	I think learners should tackle tasks.	0	0	1	3	2
5	T	I think good mathematical tasks are difficult to design.	0	1	3	2	0
6	T	I think good mathematical tasks have unforeseen learning outcomes.	0	2	1	3	0
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	1	5	0	0	0
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	3	3	0	0	0
9	T	I think tasks are coursework in disguise.	2	2	2	0	0
10	T	I think planning mathematical tasks is time consuming.	0	2	2	1	1
11	T	I think mathematical tasks take up too much teaching time.	1	3	2	0	0
12	T	I think mathematical tasks motivate learners.	0	0	1	5	0
13	T	I think designing mathematical tasks is a complex exercise.	0	1	2	2	1
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	2	2	2	0	0
15	T	I think mathematical tasks do not easily fit the current lesson structure	0	3	1	1	1
16	P	I think learners feel the 3 part lesson supports their learning.	0	2	1	3	0
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	0	2	0	4	0
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	1	3	1	1	0
19	P	I think learners should be informed as to the learning objective when tackling tasks.	0	3	2	1	0
20	P	I think learners should learn mathematics through discussing their ideas.	0	0	0	5	1
21	P	I think learner to learner dialogue (about mathematics) is important.	0	0	1	3	2
22	P	I think learners should compare and share their solutions.	0	0	1	2	3
23	P	I think learners should mainly work in pairs or groups.	0	1	3	1	1
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	0	1	1	3	1
25	T	I think it is easy to differentiate tasks for pupils	0	4	0	1	1
26	T	I think differentiating tasks for pupils is a good idea.	1	0	0	3	2

Table 4.9j – Frequency Table - Subject Leaders (n = 3)

Question Number	Category	Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	0	1	1	1	0
2	S	I think learners gain mathematical insight from practising skills.	0	2	1	0	0
3	T	I think learners should mainly work on their own when practising skills.	0	1	2	0	0
4	T	I think learners should tackle tasks.	0	0	1	2	0
5	T	I think good mathematical tasks are difficult to design.	1	1	1	0	0
6	T	I think good mathematical tasks have unforeseen learning outcomes.	0	1	1	1	0
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	2	1	0	0	0
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	0	3	0	0	0
9	T	I think tasks are coursework in disguise.	0	2	1	0	0
10	T	I think planning mathematical tasks is time consuming.	0	1	2	0	0
11	T	I think mathematical tasks take up too much teaching time.	1	0	1	0	1
12	T	I think mathematical tasks motivate learners.	0	0	2	0	1
13	T	I think designing mathematical tasks is a complex exercise.	1	1	1	0	0
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	2	1	0	0	0
15	T	I think mathematical tasks do not easily fit the current lesson structure	1	0	1	1	0
16	P	I think learners feel the 3 part lesson supports their learning.	0	1	2	0	0
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	0	1	2	0	0
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	0	3	0	0	0
19	P	I think learners should be informed as to the learning objective when tackling tasks.	0	2	0	1	0
20	P	I think learners should learn mathematics through discussing their ideas.	0	0	2	0	1
21	P	I think learner to learner dialogue (about mathematics) is important.	0	0	0	2	1
22	P	I think learners should compare and share their solutions.	0	0	0	2	1
23	P	I think learners should mainly work in pairs or groups.	0	0	0	2	1
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	0	1	0	2	0
25	T	I think it is easy to differentiate tasks for pupils	0	0	2	0	1
26	T	I think differentiating tasks for pupils is a good idea.	0	1	0	1	1

Table 4.9.k - Frequency Table PGCE subject mentors (n = 21)

Question Number	Category	Text	Almost Never	Occasionally	About half of the time	Most of the time	Almost always
1	S	I think learners should spend time in every lesson practising mathematics skills.	0	1	3	8	9
2	S	I think learners gain mathematical insight from practising skills.	0	2	7	7	5
3	T	I think learners should mainly work on their own when practising skills.	0	1	13	6	1
4	T	I think learners should tackle tasks.	0	6	3	9	3
5	T	I think good mathematical tasks are difficult to design.	2	5	11	3	0
6	T	I think good mathematical tasks have unforeseen learning outcomes.	0	6	3	5	7
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	9	9	1	1	1
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	2	19	0	0	0
9	T	I think tasks are coursework in disguise.	6	9	4	2	0
10	T	I think planning mathematical tasks is time consuming.	0	9	5	7	0
11	T	I think mathematical tasks take up too much teaching time.	1	10	5	5	0
12	T	I think mathematical tasks motivate learners.	1	4	5	9	2
13	T	I think designing mathematical tasks is a complex exercise.	0	10	5	5	1
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	1	8	9	5	1
15	T	I think mathematical tasks do not easily fit the current lesson structure	4	8	6	2	1
16	P	I think learners feel the 3 part lesson supports their learning.	2	4	2	10	3
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	1	2	4	11	3
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	3	4	7	6	1
19	P	I think learners should be informed as to the learning objective when tackling tasks.	1	2	4	7	7
20	P	I think learners should learn mathematics through discussing their ideas.	0	2	7	7	5
21	P	I think learner to learner dialogue (about mathematics) is important.	0	1	4	7	9
22	P	I think learners should compare and share their solutions.	0	0	2	6	13
23	P	I think learners should mainly work in pairs or groups.	0	5	12	3	1
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	0	1	7	9	4
25	T	I think it is easy to differentiate tasks for pupils	0	5	6	8	2
26	T	I think differentiating tasks for pupils is a good idea.	0	2	0	7	12

Table 4.9l – Statistics Table 1

(Mean, Mode and Standard Deviation) All Respondents (n = 201)

Qu	Cat	Text	Mean	Mode	Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	3.69	4	0.8452
2	S	I think learners gain mathematical insight from practising skills.	3.29	4	1.0278
3	S	I think learners should mainly work on their own when practising skills.	2.53	2	0.8368
4	T	I think learners should tackle tasks.	3.85	4	0.8761
5	T	I think good mathematical tasks are difficult to design.	3.06	2	0.9728
6	T	I think good mathematical tasks have unforeseen learning outcomes.	3.43	4	0.9519
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	2.01	2	0.9055
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	2.05	2	0.6573
9	T	I think tasks are coursework in disguise.	1.84	2	0.8970
10	T	I think planning mathematical tasks is time consuming.	3.24	4	1.0013
11	T	I think mathematical tasks take up too much teaching time.	2.10	2	0.9056
12	T	I think mathematical tasks motivate learners.	3.80	4	0.8679
13	T	I think designing mathematical tasks is a complex exercise.	3.26	4	1.0125
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	2.34	2	0.8812
15	T	I think mathematical tasks do not easily fit the current lesson structure	2.59	2	1.0553
16	P	I think learners feel the 3 part lesson supports their learning.	3.16	4	1.0510
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	3.45	4	0.8652
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2.70	2	0.9861
19	P	I think learners should be informed as to the learning objective when tackling tasks.	3.58	4	1.0702
20	P	I think learners should learn mathematics through discussing their ideas.	4.14	4	0.8310
21	P	I think learner to learner dialogue (about mathematics) is important.	4.39	5	0.7339
22	P	I think learners should compare and share their solutions.	4.46	5	0.6553
23	P	I think learners should mainly work in pairs or groups.	3.41	3	0.7764
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	3.57	3	0.9202
25	T	I think it is easy to differentiate tasks for pupils	2.65	2	0.9477
26	T	I think differentiating tasks for pupils is a good idea.	4.44	5	0.7331

Table 4.9m - Statistics Table 2

(Mean, Mode and Standard Deviation) All Respondents aged 30 years and younger (n = 155)

Qu	Cat	Text	Mean	Mode	Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	3.72	4	0.8260
2	S	I think learners gain mathematical insight from practising skills.	3.37	4	1.0066
3	S	I think learners should mainly work on their own when practising skills.	2.57	3	0.8219
4	T	I think learners should tackle tasks.	3.89	4	0.8495
5	T	I think good mathematical tasks are difficult to design.	3.07	2	0.9744
6	T	I think good mathematical tasks have unforeseen learning outcomes.	3.47	4	0.9419
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	2.01	2	0.9669
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	2.05	2	0.6678
9	T	I think tasks are coursework in disguise.	1.80	2	0.8634
10	T	I think planning mathematical tasks is time consuming.	3.15	3	1.0051
11	T	I think mathematical tasks take up too much teaching time.	2.10	2	0.8986
12	T	I think mathematical tasks motivate learners.	3.79	4	0.8900
13	T	I think designing mathematical tasks is a complex exercise.	3.20	4	1.0282
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	2.35	2	0.8797
15	T	I think mathematical tasks do not easily fit the current lesson structure	2.67	2	1.0758
16	P	I think learners feel the 3 part lesson supports their learning.	3.16	4	1.0720
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	3.45	4	0.8229
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2.77	2	0.9707
19	P	I think learners should be informed as to the learning objective when tackling tasks.	3.57	4	1.0687
20	P	I think learners should learn mathematics through discussing their ideas.	4.17	4	0.7881
21	P	I think learner to learner dialogue (about mathematics) is important.	4.39	5	0.7061
22	P	I think learners should compare and share their solutions.	4.46	5	0.6471
23	P	I think learners should mainly work in pairs or groups.	3.39	3	0.8018
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	3.57	3	0.8897
25	T	I think it is easy to differentiate tasks for pupils	2.64	2	0.9729
26	T	I think differentiating tasks for pupils is a good idea.	4.47	5	0.7054

Table 4.9n - Statistics Table 3

(Mean, Mode and Standard Deviation. All respondents aged older than 30 years (n = 46)

Qu	Cat	Text	Mean	Mode	Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	3.59	4	0.9086
2	S	I think learners gain mathematical insight from practising skills.	3.02	3	1.0644
3	S	I think learners should mainly work on their own when practising skills.	2.41	2	0.8838
4	T	I think learners should tackle tasks.	3.72	4	0.9583
5	T	I think good mathematical tasks are difficult to design.	3.02	2	0.9773
6	T	I think good mathematical tasks have unforeseen learning outcomes.	3.28	4	0.9812
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	2.00	2	0.6667
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	2.09	2	0.6263
9	T	I think tasks are coursework in disguise.	1.98	2	0.9998
10	T	I think planning mathematical tasks is time consuming.	3.54	4	0.9359
11	T	I think mathematical tasks take up too much teaching time.	2.09	2	0.9387
12	T	I think mathematical tasks motivate learners.	3.83	4	0.7973
13	T	I think designing mathematical tasks is a complex exercise.	3.48	4	0.9366
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	2.33	2	0.8958
15	T	I think mathematical tasks do not easily fit the current lesson structure	2.30	2	0.9397
16	P	I think learners feel the 3 part lesson supports their learning.	3.15	4	0.9881
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	3.46	4	1.0046
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2.43	2	1.0034
19	P	I think learners should be informed as to the learning objective when tackling tasks.	3.59	4	1.0868
20	P	I think learners should learn mathematics through discussing their ideas.	4.04	4	0.9651
21	P	I think learner to learner dialogue (about mathematics) is important.	4.39	5	0.8294
22	P	I think learners should compare and share their solutions.	4.46	5	0.6898
23	P	I think learners should mainly work in pairs or groups.	3.46	3	0.6898
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	3.54	3	1.0265
25	T	I think it is easy to differentiate tasks for pupils	2.70	3	0.8659
26	T	I think differentiating tasks for pupils is a good idea.	4.93	5	0.8180

Table 4.9o - Statistics Table 4

(Mean, Mode and Standard Deviation). All Male trainee respondents (n = 79)

Qu	Cat	Text	Mean	Mode	Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	3.63	4	0.7709
2	S	I think learners gain mathematical insight from practising skills.	3.19	4	1.0010
3	S	I think learners should mainly work on their own when practising skills.	2.62	2	0.8213
4	T	I think learners should tackle tasks.	3.76	4	0.8506
5	T	I think good mathematical tasks are difficult to design.	3.10	2	1.0076
6	T	I think good mathematical tasks have unforeseen learning outcomes.	3.54	4	0.9170
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	2.23	2	0.8908
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	2.20	2	0.7229
9	T	I think tasks are coursework in disguise.	1.86	2	0.9162
10	T	I think planning mathematical tasks is time consuming.	3.37	3	1.0150
11	T	I think mathematical tasks take up too much teaching time.	2.24	2	0.9768
12	T	I think mathematical tasks motivate learners.	3.63	4	0.9361
13	T	I think designing mathematical tasks is a complex exercise.	3.29	4	0.9891
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	2.39	2	0.9257
15	T	I think mathematical tasks do not easily fit the current lesson structure	2.62	2	1.0896
16	P	I think learners feel the 3 part lesson supports their learning.	2.94	4	1.1361
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	3.47	4	0.8448
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2.90	2	0.9687
19	P	I think learners should be informed as to the learning objective when tackling tasks.	3.58	4	1.1724
20	P	I think learners should learn mathematics through discussing their ideas.	3.95	4	0.9323
21	P	I think learner to learner dialogue (about mathematics) is important.	4.22	4	0.7624
22	P	I think learners should compare and share their solutions.	4.30	4	0.6671
23	P	I think learners should mainly work in pairs or groups.	3.35	3	0.8012
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	3.68	4	0.8850
25	T	I think it is easy to differentiate tasks for pupils	2.51	2	0.9591
26	T	I think differentiating tasks for pupils is a good idea.	4.42	5	0.7443

Table 4.9p - Statistics Table 5

(Mean, Mode and Standard Deviation). All Female trainee respondents (n = 122)

Qu	Cat	Text	Mean	Mode	Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	3.73	4	0.8911
2	S	I think learners gain mathematical insight from practising skills.	3.35	4	1.0438
3	S	I think learners should mainly work on their own when practising skills.	2.48	2	0.8451
4	T	I think learners should tackle tasks.	3.91	4	0.8908
5	T	I think good mathematical tasks are difficult to design.	3.03	3	0.9529
6	T	I think good mathematical tasks have unforeseen learning outcomes.	3.35	4	0.9700
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	1.87	2	0.8903
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	1.96	2	0.5947
9	T	I think tasks are coursework in disguise.	1.83	2	0.8879
10	T	I think planning mathematical tasks is time consuming.	3.16	4	0.9877
11	T	I think mathematical tasks take up too much teaching time.	2.01	2	0.8479
12	T	I think mathematical tasks motivate learners.	3.90	4	0.8071
13	T	I think designing mathematical tasks is a complex exercise.	3.25	4	1.0310
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	2.31	2	0.8536
15	T	I think mathematical tasks do not easily fit the current lesson structure	2.57	2	1.0364
16	P	I think learners feel the 3 part lesson supports their learning.	3.30	4	0.9697
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	3.43	4	0.8813
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2.57	2	0.9790
19	P	I think learners should be informed as to the learning objective when tackling tasks.	3.57	4	1.0034
20	P	I think learners should learn mathematics through discussing their ideas.	4.26	4	0.7363
21	P	I think learner to learner dialogue (about mathematics) is important.	4.50	5	0.6953
22	P	I think learners should compare and share their solutions.	4.56	5	0.6305
23	P	I think learners should mainly work in pairs or groups.	3.44	3	0.7611
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	3.49	3	0.9381
25	T	I think it is easy to differentiate tasks for pupils	2.75	3	0.9321
26	T	I think differentiating tasks for pupils is a good idea.	4.45	5	0.7284

Table 4.9q - Statistics Table 6

(Mean, Mode and Standard Deviation). All trainees respondents with Mathematics degrees (n = 96)

Qu	Cat	Text	Mean	Mode	Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	2.00	1	1.4216
2	S	I think learners gain mathematical insight from practising skills.	3.61	4	0.8988
3	S	I think learners should mainly work on their own when practising skills.	3.18	2	1.0857
4	T	I think learners should tackle tasks.	3.00	3	1.0662
5	T	I think good mathematical tasks are difficult to design.	3.67	4	1.0122
6	T	I think good mathematical tasks have unforeseen learning outcomes.	3.19	3	0.9549
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	2.97	2	1.1281
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	1.92	2	0.7769
9	T	I think tasks are coursework in disguise.	1.92	2	0.7351
10	T	I think planning mathematical tasks is time consuming.	2.32	2	1.1378
11	T	I think mathematical tasks take up too much teaching time.	2.78	2	1.0682
12	T	I think mathematical tasks motivate learners.	2.54	2	1.1958
13	T	I think designing mathematical tasks is a complex exercise.	3.80	4	0.8780
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	2.88	3	1.0285
15	T	I think mathematical tasks do not easily fit the current lesson structure	2.25	2	0.9515
16	P	I think learners feel the 3 part lesson supports their learning.	2.66	2	1.0841
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	3.31	4	0.9212
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	3.29	4	0.8816
19	P	I think learners should be informed as to the learning objective when tackling tasks.	3.07	4	1.2333
20	P	I think learners should learn mathematics through discussing their ideas.	3.73	4	1.0103
21	P	I think learner to learner dialogue (about mathematics) is important.	4.18	4	0.7947
22	P	I think learners should compare and share their solutions.	4.34	5	0.8056
23	P	I think learners should mainly work in pairs or groups.	4.08	5	0.9366
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	3.45	3	0.8062
25	T	I think it is easy to differentiate tasks for pupils	3.18	3	1.0259
26	T	I think differentiating tasks for pupils is a good idea.	3.32	4	1.2773

Table 4.9r - Statistics Table 7

(Mean, Mode and Standard Deviation). All trainees respondents without a Mathematics degrees; (n = 105)

Qu	Cat	Text	Mean	Mode	Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	3.58	4	0.8856
2	S	I think learners gain mathematical insight from practising skills.	3.21	4	1.0162
3	S	I think learners should mainly work on their own when practising skills.	2.48	2	0.8097
4	T	I think learners should tackle tasks.	3.79	4	0.8626
5	T	I think good mathematical tasks are difficult to design.	3.06	2	0.9887
6	T	I think good mathematical tasks have unforeseen learning outcomes.	3.40	4	0.9668
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	2.12	2	0.9776
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	2.09	2	0.6521
9	T	I think tasks are coursework in disguise.	1.85	1	0.9070
10	T	I think planning mathematical tasks is time consuming.	3.28	4	1.0423
11	T	I think mathematical tasks take up too much teaching time.	2.21	2	0.9271
12	T	I think mathematical tasks motivate learners.	3.69	4	0.9231
13	T	I think designing mathematical tasks is a complex exercise.	3.29	4	1.0443
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	2.46	2	0.8663
15	T	I think mathematical tasks do not easily fit the current lesson structure	2.74	2	1.0833
16	P	I think learners feel the 3 part lesson supports their learning.	3.12	4	1.0804
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	3.42	4	0.8856
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2.70	2	1.0296
19	P	I think learners should be informed as to the learning objective when tackling tasks.	3.48	3	1.0750
20	P	I think learners should learn mathematics through discussing their ideas.	4.19	4	0.7978
21	P	I think learner to learner dialogue (about mathematics) is important.	4.45	5	0.6648
22	P	I think learners should compare and share their solutions.	4.48	5	0.6060
23	P	I think learners should mainly work in pairs or groups.	3.43	3	0.7449
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	3.60	3	0.9364
25	T	I think it is easy to differentiate tasks for pupils	2.65	2	0.8768
26	T	I think differentiating tasks for pupils is a good idea.	4.47	5	0.6943

Table 4.9s - Statistics Table 8

(Mean, Mode and Standard Deviation). Trainee respondents from Study School; (n = 6). This data was before the trainees were in post at the school (ie whilst they were training)

Qu	Cat	Text	Mean	Mode	Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	3.67	4	1.0328
2	S	I think learners gain mathematical insight from practising skills.	3.50	3	1.0488
3	S	I think learners should mainly work on their own when practising skills.	2.33	2	1.0328
4	T	I think learners should tackle tasks.	3.83	4	1.1690
5	T	I think good mathematical tasks are difficult to design.	2.67	2	0.8165
6	T	I think good mathematical tasks have unforeseen learning outcomes.	3.50	3	0.8367
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	2.00	2	1.0954
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	2.00	2	0.6325
9	T	I think tasks are coursework in disguise.	1.67	2	0.5164
10	T	I think planning mathematical tasks is time consuming.	2.50	2	0.8367
11	T	I think mathematical tasks take up too much teaching time.	1.67	2	0.5164
12	T	I think mathematical tasks motivate learners.	4.00	4	0.6325
13	T	I think designing mathematical tasks is a complex exercise.	2.83	3	0.7528
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	2.00	2	0.6325
15	T	I think mathematical tasks do not easily fit the current lesson structure	2.50	2	0.8367
16	P	I think learners feel the 3 part lesson supports their learning.	3.17	2	1.1690
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	3.50	3	0.5477
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	3.00	2	0.8944
19	P	I think learners should be informed as to the learning objective when tackling tasks.	3.83	4	1.1690
20	P	I think learners should learn mathematics through discussing their ideas.	4.33	5	0.8165
21	P	I think learner to learner dialogue (about mathematics) is important.	4.50	4	0.5477
22	P	I think learners should compare and share their solutions.	4.50	4	0.5477
23	P	I think learners should mainly work in pairs or groups.	4.00	4	0.6325
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	3.33	3	1.0328
25	T	I think it is easy to differentiate tasks for pupils	3.00	3	0.6325
26	T	I think differentiating tasks for pupils is a good idea.	4.50	4	0.5477

Table 4.9t - Statistics Table 9

(Mean, Mode and Standard Deviation). Trainee respondents from Study School; (n = 6).This data was after the trainees were in post at the school (after qualification).

Qu	Cat	Text	Mean	Mode	Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	3.83	4	0.9832
2	S	I think learners gain mathematical insight from practising skills.	3.17	3	0.7528
3	S	I think learners should mainly work on their own when practising skills.	2.67	3	1.0328
4	T	I think learners should tackle tasks.	4.17	4	0.7528
5	T	I think good mathematical tasks are difficult to design.	3.17	3	0.7528
6	T	I think good mathematical tasks have unforeseen learning outcomes.	3.17	4	0.9832
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	1.83	2	0.4082
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	1.50	1	0.5477
9	T	I think tasks are coursework in disguise.	2.33	1	1.3663
10	T	I think planning mathematical tasks is time consuming.	3.17	2	1.1690
11	T	I think mathematical tasks take up too much teaching time.	2.17	2	0.7528
12	T	I think mathematical tasks motivate learners.	3.83	4	0.4082
13	T	I think designing mathematical tasks is a complex exercise.	3.50	3	1.0488
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	2.00	1	0.8944
15	T	I think mathematical tasks do not easily fit the current lesson structure	3.00	2	1.2649
16	P	I think learners feel the 3 part lesson supports their learning.	3.17	4	0.9832
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	3.33	4	1.0328
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2.33	2	1.0328
19	P	I think learners should be informed as to the learning objective when tackling tasks.	2.83	2	1.1690
20	P	I think learners should learn mathematics through discussing their ideas.	4.17	4	0.4082
21	P	I think learner to learner dialogue (about mathematics) is important.	4.17	4	0.7528
22	P	I think learners should compare and share their solutions.	4.33	5	0.8165
23	P	I think learners should mainly work in pairs or groups.	3.33	3	1.0328
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	3.67	4	1.0328
25	T	I think it is easy to differentiate tasks for pupils	2.83	2	1.3292
26	T	I think differentiating tasks for pupils is a good idea.	3.83	4	1.4720

Table 4.9u - Statistics Table 10

(Mean, Mode and Standard Deviation). Subject Leaders from Study School; (n = 3)

Qu	Cat	Text	Mean	Mode	Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	3.00	2	1.0000
2	S	I think learners gain mathematical insight from practising skills.	2.33	2	0.5774
3	S	I think learners should mainly work on their own when practising skills.	2.67	3	0.5774
4	T	I think learners should tackle tasks.	3.67	4	0.5774
5	T	I think good mathematical tasks are difficult to design.	2.33	1	1.5275
6	T	I think good mathematical tasks have unforeseen learning outcomes.	3.00	2	1.0000
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	1.33	1	0.5774
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	2.00	2	0.0000
9	T	I think tasks are coursework in disguise.	2.33	2	0.5774
10	T	I think planning mathematical tasks is time consuming.	3.33	4	1.1547
11	T	I think mathematical tasks take up too much teaching time.	2.67	1	1.5275
12	T	I think mathematical tasks motivate learners.	3.67	3	1.1547
13	T	I think designing mathematical tasks is a complex exercise.	2.00	1	1.0000
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	1.33	1	0.5774
15	T	I think mathematical tasks do not easily fit the current lesson structure	2.67	1	1.5275
16	P	I think learners feel the 3 part lesson supports their learning.	2.67	3	0.5774
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	3.33	4	1.1547
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2.00	2	0.0000
19	P	I think learners should be informed as to the learning objective when tackling tasks.	2.67	2	1.1547
20	P	I think learners should learn mathematics through discussing their ideas.	3.67	3	1.1547
21	P	I think learner to learner dialogue (about mathematics) is important.	4.33	4	0.5774
22	P	I think learners should compare and share their solutions.	4.33	4	0.5774
23	P	I think learners should mainly work in pairs or groups.	3.33	3	0.5774
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	3.33	4	1.1547
25	T	I think it is easy to differentiate tasks for pupils	3.67	3	1.1547
26	T	I think differentiating tasks for pupils is a good idea.	3.67	2	1.5275

Table 4.9v - Statistics Table 11

(Mean, Mode and Standard Deviation). PGCE Subject Mentors Respondents (n = 21)

Qu	Cat	Text	Mean	Mode	Std. Deviation
1	S	I think learners should spend time in every lesson practising mathematics skills.	4.19	5	0.8729
2	S	I think learners gain mathematical insight from practising skills.	3.71	3	0.9562
3	S	I think learners should mainly work on their own when practising skills.	3.33	3	0.6583
4	T	I think learners should tackle tasks.	3.43	4	1.0757
5	T	I think good mathematical tasks are difficult to design.	2.71	3	0.8452
6	T	I think good mathematical tasks have unforeseen learning outcomes.	3.62	5	1.2440
7	T	I think good mathematical tasks can lead learners along unproductive pathways.	1.86	1	1.0623
8	T	I think mathematical tasks can lead learners to incorrect mathematical conclusions.	1.90	2	0.3008
9	T	I think tasks are coursework in disguise.	2.10	2	0.9437
10	T	I think planning mathematical tasks is time consuming.	2.90	2	0.8891
11	T	I think mathematical tasks take up too much teaching time.	2.67	2	0.9129
12	T	I think mathematical tasks motivate learners.	3.33	4	1.0646
13	T	I think designing mathematical tasks is a complex exercise.	2.86	2	0.9636
14	T	I think designing mathematical tasks is not necessary as other resources are readily available.	2.86	2	1.0142
15	T	I think mathematical tasks do not easily fit the current lesson structure	2.43	2	1.0757
16	P	I think learners feel the 3 part lesson supports their learning.	3.38	4	1.2440
17	P	I think learners feel confident and secure if I use one or two teaching approaches.	3.62	4	1.0235
18	P	I think learners feel confident and secure if I only use one or two teaching approaches.	2.90	3	1.1360
19	P	I think learners should be informed as to the learning objective when tackling tasks.	3.81	4	1.1670
20	P	I think learners should learn mathematics through discussing their ideas.	3.71	3	0.9562
21	P	I think learner to learner dialogue (about mathematics) is important.	4.14	5	0.9103
22	P	I think learners should compare and share their solutions.	4.52	5	0.6796
23	P	I think learners should mainly work in pairs or groups.	3.00	3	0.7746
24	S	I think exercises which gradually increase in difficulty build learners mathematical confidence better than tasks.	3.76	4	0.8309
25	T	I think it is easy to differentiate tasks for pupils	3.33	4	0.9661
26	T	I think differentiating tasks for pupils is a good idea.	4.38	5	0.9207

Table 5.2.1 - Sample of pupil responses from the starter activity.
Each line in the table is the outcomes from a pair of pupils.

Numerator is the top number. Denominator is the bottom number.	1/4 of 1/2 would mean 1/4 times(x) 1/2	On some fractions we must do the same to the numerator and the denominator	Names of fractions: Improper, Equivalent, Top Heavy	Part of a whole / a portion of something	
Numerators = Top number, Denominator = Bottom number	Improper fractions	That you can have top heavy	You can have quarter(sp) and 3rds		
There's a numerator and a denominator	The smiley face method / Cross multiple(y) 1/2 + 1/3 = 3/6+2/6	There can be top heavy fractions / Improper fractions			
Denominator is the bottom number	Fractions can be turned into percentages	Improper fractions = Top Heavy 7/3	Equivalent fractions 1/2 and 2/4	You can't have decimals in fractions	Numerator is the top number
They can be changed into numbers, decimals and percentages	You can add, subtract, multiply and divide a fraction	There are top heavy fractions, this is when the top numbers is larger than the bottom number eg 10/5	ordering fractions		

Fractions have got something to do with numbers	Fractions are like a decimal	All fractions are less than 1		
Simplify certain fractions	How to convert fractions	Numerator, Denominator	Types of fractions : Improper etc top heavy	
You can simplify fractions to their simplest form	Numerator is the top and the denominator is the bottom eg 1/4 - 1 numerator, 4 denominator)			
When you times or divide the number you must do the same to the denominator	All fractions have a numerator and a denominator	You can have 1 whole and some fraction like 1 2/7	You can have a top heavy numerator	
You can't have decimals in fractions	If the denominator is smaller its top heavy (improper)	Numerators and denominators make up a fraction	What you do to the bottom (denominator) you do to the top (numerator)	Denominator is most likely to be higher
Top heavy fractions 7/3	Numerator / Denominator (corrected on sheet)	1/2, 1/4, 3/4 and 1	You can add, subtract and times them.	

Numerator = Top, Denominator = Bottom	Most things can be changed into fractions or can be cut into them	When the numerator is larger than the denominator it is called an improper fraction				
You can't have decimals in fractions	An equivalent fraction is equal	Fractions are numbers	There is a top half and a bottom half in a fraction	There's a fraction used in everything	There are numerators and denominators	They are shown in shops

Colour codes for categorisation.

Misconceptions
Type of Fraction
Recall Facts
Numerator / Denominator
Actions

	7NR	7AC
Misconceptions	3 (3%)	8 (19%)
Types of fractions	52 (56%)	7 (16%)
Recalling Facts	5 (5%)	15 (35%)
Numerator / Denominator	12 (13%)	10 (23%)
Actions on Fractions	16 (17%)	1 (2%)
None of the above	5 (5%)	1 (2%)

Table 5.2.5 - Analysis of the Task for both classes

The table shows the number of questions of the type “What if” that were composed by each pair o pupils and the number of correct and incorrect responses.

Class / pair	Questions written	Correct		Incorrect
7NR/1	3	2		1
7NR/2	3	3		
7NR/3	3	3		
7NR/4	4	2		2
7NR/5	2	2		
7NR/6	3	3		
7NR/7	1			1
7NR/8	2	2		
7NR/9	1	1		
Sub Total		**18**		**4**
7AC/1	3	3		0
7AC/2	3	3		0
7AC/3	3	3		0
7AC/4	3	2		1
7AC/5	1	1		
7AC/6	2	2		
7AC/7	1	1		
7AC/8	1	1		
7AC/9	2	2		
7AC/10	2	1		1
7AC/11	1	1		
7AC/12	3	2		1
7AC/13	1	1		
Sub Total		**23**		**3**

Table 5.2.6 – Frequencies for pupil learning.
In the four stages of the two lessons combined.

	A – Teacher Input – teaching / demonstrating / explaining	B – Pupil – Pupil Dialogue	C – Pupil Reasoning	D – Interventions	E – Pupils Using the manipulatives	F – Pupil – Teacher Dialogue	G – Pupil demonstrating understanding	H – Connecting learning (eg division of numbers with division of fractions)
Exercise	30	14	14	10	18	19	13	12
Activity	34	17	13	17	12	8	14	8
Skill	25	25	10	17	20	11	11	10
Tasks	32	19	22	25	22	19	25	27

This table gives the frequencies of observed pupil learning in the four stages of the two lessons combined against the 8 criteria.